Worksheet (2)
– Solutions –

(1) $\frac{dy}{dx} = -2y^2 x \quad \Big| \cdot \frac{1}{y^2}$ ⇝ dividing by y, we are assuming $y \neq 0$

$$\int -\frac{1}{y^2}\,dy = \int 2x\,dx$$

$$\boxed{\frac{1}{y} = x^2 + C} \Rightarrow \boxed{y = \frac{1}{x^2+C}}$$

- Is $y=0$ a solution? Yes
- Is $y=0$ represented in the formula $y = \frac{1}{x^2+C}$?

  No – the function $y = \frac{1}{x^2+C}$ is in fact never 0!
- Complete answer: $\boxed{y(x) = \frac{1}{x^2+C} \text{ or } y(x) = 0}$ $\quad x \in (-\infty, \infty)$

(2) $y' = x(y-1)$

$$\frac{dy}{dx} = x(y-1) \quad \Big| \cdot \frac{1}{(y-1)}$$

$$\int \frac{1}{y-1}\,dy = \int x\,dx$$

$$\ln|y-1| = \frac{x^2}{2} + C$$

$$|y-1| = e^{x^2/2 + C} = e^C e^{x^2/2}$$

$$y-1 = \pm e^C e^{x^2/2}$$

$$y-1 = Ce^{x^2/2}$$

$$\boxed{y = 1 + Ce^{x^2/2}}$$

Did we lose a solution?

- Is $y=1$ a solution? Go back to:

$$y' = x(y-1)$$

If $y(x)=1$ for all x:

$$\left.\begin{aligned} y'(x) &= \underline{\underline{0}} \text{ for all } x \\ x(\underbrace{y(x)}_{1}-1) &= \underline{\underline{0}} \text{ for all } x \end{aligned}\right\} \Rightarrow 0=0 \text{ for all } x \ ✓$$

<u>Yes</u>, $y=1$ is a solution.

- Is $y=1$ represented in the formula $y = 1 + Ce^{x^2/2}$?

  Yes – by letting $C=0$.
- Complete answer:

$$\boxed{y = 1 + Ce^{x^2/2}} \quad x \in (-\infty, \infty)$$

③ $y' = 2x(1-y)^2$

$$\frac{dy}{dx} = 2x(1-y)^2 \quad \Big| \cdot \frac{1}{(1-y)^2}$$

$$\frac{1}{(1-y)^2}\,dy = 2x\,dx$$

$$\frac{1}{1-y} = x^2+C \Rightarrow 1-y = \frac{1}{x^2+C} \Rightarrow \boxed{y = 1 - \frac{1}{x^2+C}}$$

- Is $y=1$ a solution? Yes
- Is $y=1$ represented in the formula? No : for $y = 1 - \frac{1}{x^2+C} = 1$

  we would need $\frac{1}{x^2+C} = 0$, which never happens.

- Complete answer:

$$\boxed{y(x) = 1 - \frac{1}{x^2+C} \text{ or } y(x) = 0} \quad x \in (-\infty, \infty).$$

④ $\frac{dy}{dx} = y^2 - 4$; $y(0) = -2$.

$$\int \frac{1}{y^2-4}\,dy = \int dx \Rightarrow \frac{1}{4}\ln\left|\frac{y-2}{y+2}\right| = x + C$$

$$\frac{1}{y^2-4} = \frac{1}{(y-2)(y+2)}$$

$$= \frac{1}{4}\,\frac{(y+2)-(y-2)}{(y-2)(y+2)}$$

$$= \frac{1}{4}\left(\frac{1}{y-2} - \frac{1}{y+2}\right)$$

$$\Rightarrow \int \frac{1}{y^2-4}\,dy = \frac{1}{4}\left(\ln|y-2| - \ln|y+2|\right)$$

$$= \frac{1}{4}\ln\left|\frac{y-2}{y+2}\right|$$

$$\ln\left|\frac{y-2}{y+2}\right| = 4x + C$$

$$\left|\frac{y-2}{y+2}\right| = e^{4x+C} = e^C e^{4x}$$

$$\frac{y-2}{y+2} = \pm e^C e^{4x}$$

$$\frac{y-2}{y+2} = Ce^{4x}$$

We have the implicit solution : $\boxed{\frac{y-2}{y+2} = ce^{4x}}$

Go back to the initial condition to find c :

$$x=0;\ y=-2 \implies \frac{-4}{0} = c$$?!

<u>What went wrong</u> ?

When dividing by $(y^2-4)=(y-2)(y+2)$, we assumed implicitly that $y \neq 2$ and $y \neq -2$. So, we might have lost <u>two solutions</u>.

Are $y=2$ and $y=-2$ solutions to the ODE? Yes.

Are $y=2$ and $y=-2$ represented in the formula $\frac{y-2}{y+2} = ce^{4x}$?

It looks like $y=-2$ definitely is not but let's make sure & solve for y :

$$y-2 = ce^{4x}(y+2)$$

$$y(1-ce^{4x}) = 2(1+ce^{4x})$$

$$y = \frac{2(1+ce^{4x})}{1-ce^{4x}}$$

- If we want $y=2$, we must have : $\frac{1+ce^{4x}}{1-ce^{4x}} = 1$ for all x

$$\implies ce^{4x} = -ce^{4x} \text{ for all } x$$

$$\implies \boxed{c=0}$$

So the solution $y=2$ <u>is</u> represented, by letting $c=0$.

- If we want $y=-2$, we must have $\frac{1+ce^{4x}}{1-ce^{4x}} = -1$ for all x

$$\implies 1+\cancel{ce^{4x}} = -1+\cancel{ce^{4x}}$$

$$\implies 1=-1 \text{ (Contradiction!)}$$

So <u>no</u> value of c can give $y=-2$ in our formula.

<u>So</u> : A complete answer to solving $\frac{dy}{dx} = y^2-4$ is $\boxed{y = \frac{2(1+ce^{4x})}{1-ce^{4x}} \text{ or } y=-2}$

If we add the initial condition $y(0)=-2$, we already saw that the only possibility is $y=-2$.

$\implies$ answer to the IVP : $y=-2$.

5a) $x\frac{dy}{dx} = y^2 - y \quad ; y(0) = 1$

$$\int \frac{1}{y(y-1)}dy = \int \frac{1}{x}dx$$

$$\int \frac{1}{y(y-1)}dy = \int \frac{y-(y-1)}{y(y-1)}dy = \int \frac{1}{y-1}dy - \int \frac{1}{y}dy = \ln|y-1| - \ln|y| + c$$

$$\Rightarrow \ln\left|\frac{y-1}{y}\right| = \ln|x| + c$$

$$\Rightarrow \left|\frac{y-1}{y}\right| = |x|e^c \Rightarrow \boxed{\boxed{\frac{y-1}{y} = cx} \text{ or } \boxed{y=0}}$$

Possibly lost solutions? $y=0$ and $y=1$. But $y=1$ is represented in $\frac{y-1}{y} = cx$, for $c=0$. However, $y=0$ is not.

Initial Condition: $x=0$, $y=1 \Rightarrow \frac{1-1}{1} = c \cdot 0 \Rightarrow 0=0$

This is true for all c!

What does that mean? That all the functions of the form $\frac{y-1}{y} = cx$ go through the point $(0,1)$.

You can also solve for y:
$y-1 = cxy$
$y(1-cx) = 1$
$y = \frac{1}{1-cx}$

Does the other possible solution to this ODE, $y=0$, go through this point? No.

$\Rightarrow$ Answer to 5a: $\boxed{y = \frac{1}{1-cx} \; ; c \in \mathbb{R}}$ (infinitely many solutions)

(or, you could leave it as $\frac{y-1}{y} = cx$).

5b) $x\frac{dy}{dx} = y^2 - y \; ; \; y(1) = 1$

Put $x=1, y=1$ in $y = \frac{1}{1-cx} \Rightarrow 1 = \frac{1}{1-c} \Rightarrow 1-c = 1 \Rightarrow c=0$

$c=0 \Rightarrow y = \frac{1}{1-0}$ for all $x \Rightarrow$ Answer to 5b: $\boxed{y=1}$

5c) $x\frac{dy}{dx} = y^2 - y \; ; \; y(1) = 0$

Put $x=1, y=0$ in $y = \frac{1}{1-cx} \Rightarrow 0 = \frac{1}{1-c}$ cannot happen for any c

$\Rightarrow$ Answer to 5c: $\boxed{y=0}$