

Integration by Parts:

$$\int f'(g) = fg - \int (f)'g$$

(Comes from the Product Rule :  $(fg)' = (f)'g + f(g)'$

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①  $\int x e^{2x} dx$

want to differentiate  $x$ , b/c  $(x)' = 1$   
 $\Rightarrow$  express the other function ( $e^{2x}$ ) as a derivative:

$$\begin{aligned} \int x e^{2x} dx &= \int x \left(\frac{1}{2} e^{2x}\right)' dx = (x) \cdot \left(\frac{1}{2} e^{2x}\right) - \int (x)' \left(\frac{1}{2} e^{2x}\right) dx \\ &= \frac{1}{2} x e^{2x} - \frac{1}{2} \int e^{2x} dx \\ &= \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} + c \end{aligned}$$

②  $\int x \sin x dx$

$$\begin{aligned} &= \int x (-\cos x)' dx \\ &= -x \cos x + \int \cos x dx \\ &= -x \cos x + \sin x + c \end{aligned}$$

③  $\int x \ln x dx$

$$\begin{aligned} &= \int \left(\frac{x^2}{2}\right)' \ln x dx \\ &= \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} dx \\ &= \frac{x^2}{2} \ln x - \frac{x^2}{4} + c \end{aligned}$$

$$\textcircled{4} \int x^2 e^{-x} dx$$

$$= \int x^2 (-e^{-x})' dx$$

$$= -x^2 e^{-x} + \int 2x e^{-x} dx$$

$$= -x^2 e^{-x} - 2 \int x (e^{-x})' dx$$

↪ by parts again

$$= -x^2 e^{-x} - 2x e^{-x} + 2 \int e^{-x} dx$$

$$= \boxed{-x^2 e^{-x} - 2x e^{-x} - 2e^{-x} + c}$$

$$\textcircled{5} \int \arctan(x) dx$$

$$= \int (x)' \arctan(x) dx$$

$$= x \arctan(x) - \int x \cdot \frac{1}{1+x^2} dx$$

$$= \boxed{x \arctan(x) - \frac{1}{2} \ln(1+x^2) + c}$$

$$\textcircled{6} \int e^x \sin(x) dx$$

$$I := \int e^x \sin(x) dx$$

$$= \int (e^x)' \sin(x) dx = e^x \sin(x) - \int e^x \cos(x) dx$$

$$= e^x \sin(x) - \int (e^x)' \cos(x) dx$$

$$= e^x \sin(x) - e^x \cos(x) - \underbrace{\int e^x \sin(x) dx}_I$$

↪ by parts again

$$\Rightarrow 2I = e^x \sin(x) - e^x \cos(x)$$

$$\Rightarrow \boxed{I = \frac{1}{2} e^x (\sin(x) - \cos(x))}$$

$$\textcircled{7} \int \frac{(\ln x)^2}{x^3} dx$$

Solution I (Integration by Parts):

$$\begin{aligned} \int \frac{(\ln x)^2}{x^3} dx &= \int (\ln x)^2 \cdot \left(\frac{-1}{2x^2}\right)' dx \\ &= -\frac{1}{2x^2} (\ln x)^2 + \int \frac{1}{2x^2} \cdot 2(\ln x) \cdot \frac{1}{x} dx \\ &= -\frac{1}{2x^2} (\ln x)^2 + \int \frac{1}{x^3} (\ln x) dx \\ &= -\frac{1}{2x^2} (\ln x)^2 + \int \left(\frac{-1}{2x^2}\right)' (\ln x) dx \\ &= -\frac{1}{2x^2} (\ln x)^2 - \frac{1}{2x^2} (\ln x) + \int \frac{1}{2x^2} \cdot \frac{1}{x} dx \\ &= \boxed{-\frac{1}{2x^2} (\ln x)^2 - \frac{1}{2x^2} (\ln x) - \frac{1}{4x^2} + c} \end{aligned}$$

Solution II : (Substitution + by Parts)

$$\begin{aligned} u = \ln x &\Rightarrow x = e^u \\ du = \frac{1}{x} dx &\Rightarrow dx = x du = e^u du \end{aligned} \quad \left. \vphantom{\begin{aligned} u = \ln x \\ du = \frac{1}{x} dx \end{aligned}} \right\} \Rightarrow \int \frac{u^2}{(e^u)^3} e^u du = \int u^2 e^{-2u} du$$

$$\begin{aligned} \int u^2 e^{-2u} du &= -\frac{1}{2} u^2 e^{-2u} + \int u e^{-2u} du \\ &= -\frac{1}{2} u^2 e^{-2u} - \frac{1}{2} u e^{-2u} - \frac{1}{4} e^{-2u} + c \end{aligned}$$

$$\Rightarrow \boxed{-\frac{1}{2} (\ln x)^2 (x^{-2}) - \frac{1}{2} (\ln x) (x^{-2}) - \frac{1}{4} (x^{-2}) + c}$$

$$\textcircled{8} \int \frac{1}{x-2} dx = \ln|x-2| + C$$

$$\textcircled{9} \int \frac{x^2 - 2x}{x^3 - 3x^2 + 1} dx = \frac{1}{3} \ln|x^3 - 3x^2 + 1| + C$$

Remark:  $(x^3 - 3x^2 + 1)' = 3x^2 - 6x = 3(x^2 - 2x)$

$$\textcircled{10} \int \frac{s+2}{s-3} ds = \int \frac{(s-3)+5}{s-3} ds = \int \left(1 + \frac{5}{s-3}\right) ds = s + 5 \ln|s-3| + C$$

$$\textcircled{11} \int \frac{x}{x^2 + 2x - 3} dx = \int \frac{x}{(x-1)(x+3)} dx$$

Partial Fraction Decomposition:  $\frac{x}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3}$

$$x = Ax + 3A + Bx - B$$

$$x = (A+B)x + (3A-B)$$

$$\begin{cases} A+B=1 \\ 3A-B=0 \Rightarrow B=3A \end{cases}$$

$$\Rightarrow 4A=1 \Rightarrow A=1/4; B=3/4$$

$$\begin{aligned} \int \frac{x}{(x-1)(x+3)} dx &= \frac{1}{4} \int \frac{1}{x-1} dx + \frac{3}{4} \int \frac{1}{x+3} dx \\ &= \frac{1}{4} \ln|x-1| + \frac{3}{4} \ln|x+3| + C \end{aligned}$$

$$\textcircled{12} \int \frac{3+x^2}{x} dx = \int \left(\frac{3}{x} + x\right) dx = 3 \ln|x| + \frac{x^2}{2} + C$$

$$\begin{aligned} \textcircled{13} \int \frac{1}{x(3+\ln x)} dx &= \int \frac{1}{3+u} du \quad [u = \ln x] \\ &= \ln|3+u| + C = \ln|3+\ln x| + C \end{aligned}$$

$$\textcircled{14} \int \frac{x^2}{1+x^2} dx = \int \frac{(1+x^2)-1}{1+x^2} dx = \int \left(1 - \frac{1}{1+x^2}\right) dx = x - \arctan x + C$$