Existence & Uniqueness of Solutions to ODEs

- Recall initial value problems (IVPs):

(n^{th} order) IVP:

Solve: $$\frac{d^n y}{dx^n} = f(x, y, y', \dots, y^{(n-1)})$$

Subject to: $$y(x_0) = y_0\ ;\ y'(x_0) = y_1\ ;\ \dots\ ;\ y^{(n-1)}(x_0) = y_{n-1}$$

- Solving these usually involves:
  (1). Finding an n-parameter family of solutions (**general solution**)
  (2). Using the initial conditions at x_0 to determine the n constants (**particular solution**).

- The resulting particular solution is defined on some interval I that contains the initial point x_0 (**interval of validity**) (aka "interval of definition", "interval of existence").

  - **Example 1:** The 1-parameter family $\boxed{y = \frac{2}{x^2+C}}$ is a solution to the ODE $\boxed{y' + xy^2 = 0}$. Imposing the initial condition $y(0) = -2$, we obtain the particular solution

$$\boxed{y = \frac{2}{x^2-1}}$$

→ Considered as a function, its domain is $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

→ Considered as a solution to the ODE $\boxed{y' + xy^2 = 0}$, the interval of validity can be taken to be any interval on which $y(x)$ is defined and differentiable;
  - The largest possible ones: $(-\infty, -1)$; $(-1, 1)$; $(1, \infty)$.

→ Considered as a solution to the IVP $\boxed{y' + xy^2 = 0\ ;\ y(0) = -2}$, the interval of validity can be taken to be any interval on which $y(x)$ is defined, differentiable and contains the initial point $x = 0$.
  - The largest possible one: $(-1, 1)$.

Existence & Uniqueness of 1st Order IVPs

- Consider a first order IVP:

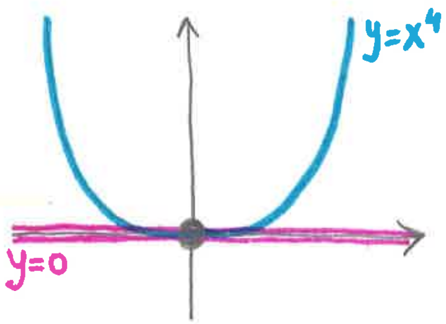
$$\boxed{\frac{dy}{dx} = f(x,y) \; ; \; y(x_0) = y_0} \qquad (*)$$

- Two important questions that arise:
  1). **Existence:** Does the equation $y' = f(x,y)$ have any solutions? If so, do any of the solution curves pass through (x_0, y_0)?

  2). **Uniqueness:** Out of all the solution curves to $y' = f(x,y)$, is there exactly one that passes through (x_0, y_0)?

- Remark : An IVP can have more than one solution

- **Example 2** : Both $y=0$ and $y=x^4$ are solutions to the IVP:



$$\boxed{\frac{dy}{dx} = 4x\sqrt{y} \; ; \; y(0) = 0}$$

(these are two solution curves which both pass through the point $(0,0)$).

- The following theorem provides sufficient conditions to guarantee existence & uniqueness of solutions to a first order IVP.

THEOREM 1:

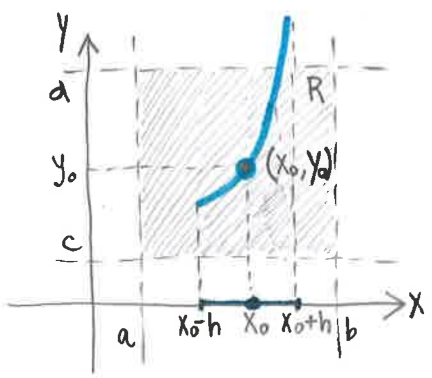
Consider the first order IVP:

$$y' = f(x,y)\ ;\ y(x_0) = y_0 \quad (*)$$

and let R $(a<x<b\ ;\ c<y<d)$ be a rectangular region in the plane that contains the point (x_0, y_0). If:

$$\boxed{f(x,y) \text{ and } \frac{\partial f}{\partial y}(x,y)}$$

are continuous on R, then there exists a unique solution $y(x)$ to $(*)$, defined on an interval $I = (x_0 - h, x_0 + h)$; $h > 0$, contained in (a,b).



Remark: Revisit Example 2:

$$y' = 4x\sqrt{y}\ ;\ y(1) = 4$$

(we changed the initial condition).
The equation can be written in the form

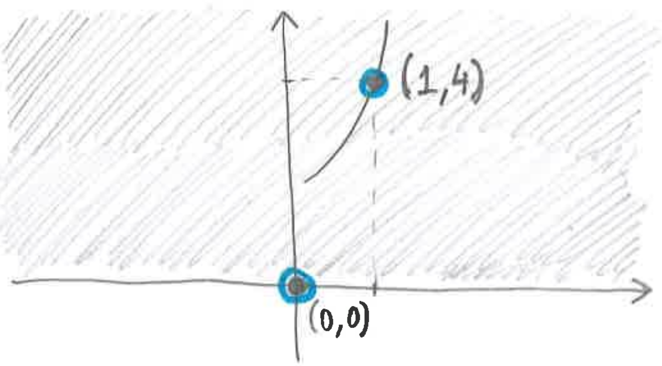
$$y' = f(x,y)$$

with $f(x,y) = 4x\sqrt{y}$

Then:

$$\boxed{\frac{\partial f}{\partial y}(x,y) = \frac{2x}{\sqrt{y}}}$$

Note that both $f(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$ are continuous on $\mathbb{R} \times (0,\infty)$, the upper half-plane.



The point $(1,4)$ is contained in this (infinite) rectangle, so Theorem 1 guarantees that the IVP

$$y' = 4x\sqrt{y}\ ;\ y(1) = 4$$

has a unique solution.

(Check using separation of variables that this unique solution is $\sqrt{y} = x^2 + 1$, or $y = (x^2+1)^2$).

Necessary vs. Sufficient Conditions

Consider two statements P and Q.

- P is called a sufficient condition for Q if $\boxed{P \to Q}$.

  Example: $\boxed{f(x) \text{ is differentiable at } x = x_0} \Rightarrow \boxed{f \text{ is continuous at } x = x_0}$

  (sufficient condition for continuity)

- P is called a necessary condition for Q if $\boxed{Q \to P}$

  Example: $\boxed{f(x) \text{ is differentiable at } x = x_0} \Rightarrow \boxed{f \text{ is continuous at } x = x_0}$

  (necessary condition for differentiability)

Note then that P is a necessary and sufficient condition for Q if $\boxed{P \leftrightarrow Q}$

- We know that "f is continuous at $x = x_0$" is a necessary condition for differentiability of f at $x = x_0$, but it is not sufficient (there are functions that are continuous at $x = x_0$ but are not differentiable at $x = x_0$; think of $f(x) = |x|$ for example).

- We saw that Theorem 1 provides sufficient conditions for the existence and uniqueness of a solution to a first order IVP

$$\boxed{\begin{array}{c} f(x,y) \text{ and } \frac{\partial f}{\partial y}(x,y) \\ \text{continuous on } R \\ (R \text{ contains } (x_0, y_0)) \end{array}} \Rightarrow \boxed{\begin{array}{c} \exists \text{ unique solution } y(x) \text{ to} \\ y' = f(x,y)\ ;\ y(x_0) = y_0 \end{array}}$$

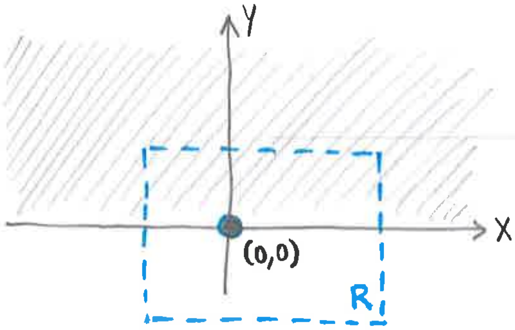
These conditions however, are not necessary - meaning that, if these conditions are not met, anything can happen!

Important point to stress: just because these conditions are not met, does not mean that there is no unique solution.

There are instances when there is, and instances where there isn't a unique solution.

- **Example 2** (again): $\boxed{y' = 4x\sqrt{y} \ ; \ y(0)=0}$ → does not satisfy Theorem 1 → does not have unique solution

We already saw that this IVP has at least two solutions ($y=0$ and $y=x^4$)
This IVP does not satisfy the conditions of Theorem 1:



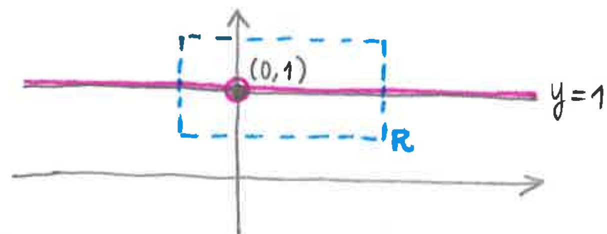
$$f(x,y) = 4x\sqrt{y} \ ; \quad \frac{\partial f}{\partial y}(x,y) = \frac{2x}{\sqrt{y}}$$

Any rectangle R containing $(0,0)$ will also contain parts of the lower half-plane, where $f(x,y)$ is not even defined.

- **Example 3**: $\boxed{y' = |y-1| \ ; \ y(0)=1}$ → does not satisfy Theorem 1 → does have a unique solution

$y' = f(x,y)$;

$$f(x,y) = |y-1| = \begin{cases} y-1, & \text{if } y \geq 1 \\ 1-y, & \text{if } y<1 \end{cases}$$



$f(x,y)$ is continuous on $\mathbb{R}^2$, but $\frac{\partial f}{\partial y}(x,y)$ does not even exist whenever $y=1$.
Since any rectangle containing $(0,1)$ will also contain part of the line $y=1$, this IVP does not satisfy the conditions of Theorem 1.

Solve the ODE: $\frac{dy}{dx} = |y-1|$

⊛ If $y(x) > 1$: $\frac{dy}{dx} = y-1 \Rightarrow \frac{1}{y-1}dy = dx \Rightarrow \ln(y-1) = x+c \Rightarrow y-1 = e^{x+c}$
$\Rightarrow \boxed{y = 1+ce^{x} \ ; \ c>0}$ (always >1)

⊛ If $y(x) < 1$: $\frac{dy}{dx} = 1-y \Rightarrow \frac{1}{1-y}dy = dx \Rightarrow -\ln(1-y) = x+c \Rightarrow \frac{1}{1-y} = e^{x+c}$
$\Rightarrow \boxed{y = 1-ce^{-x} \ ; \ c>0}$ (always <1)

⊛ If $y=1 \Rightarrow 0=0$ for all x
($y=1$ is a solution)

$\Rightarrow$ General Solution: $\boxed{[y=1+ce^{x} \ ; \ c>0] \ \& \ [y=1-ce^{-x} \ ; \ c>0] \ \& \ [y=1]}$

Find particular solution w/ $x=0, y=1$: $[1=1+c \Rightarrow c=0] \ \& \ [1=1-c \Rightarrow c=0] \ \& \ [y=1]$
$\Rightarrow \boxed{y=1}$ is the only solution to the IVP.

Remark : IVPs are characterized by their initial conditions, which are all given at the same value of $x = x_0$; for example:

$$y'' + 2y = 0 \ ; \ y(0) = 0 \ ; \ y'(0) = 0$$

Sometimes we are interested in solving higher order ODEs where the function and/or its derivatives are specified at different values of x; for example:

$$y'' + 2y = 0 \ ; \ y(0) = 0 \ ; \ y(2) = 1$$

$$y'' + 2y = 0 \ ; \ y(\pi) = 0 \ ; \ y'(\pi/2) = 1$$

$$y'' + 2y = 0 \ ; \ y'(0) = 0 \ ; \ y'(\pi) = 0$$

These are called boundary value problems (BVP) - later.

Theorem 1 has the following useful consequence for 1st order linear ODEs:

COROLLARY 1 : Existence & Uniqueness of Solutions to 1st order linear ODEs

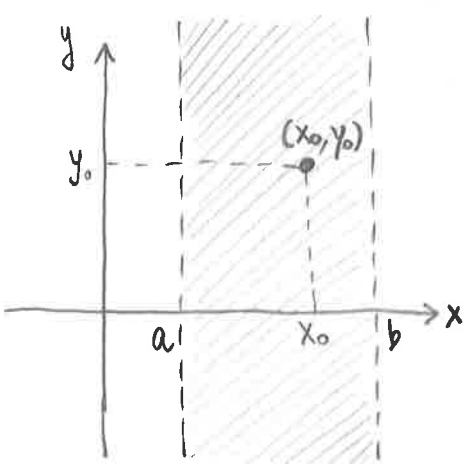
Consider the first order linear IVP:

$$y' + p(x)y = g(x) \ ; \ y(x_0) = y_0 \qquad (**)$$

and let (a,b) be an interval containing x_0. If

$$\boxed{p(x) \text{ and } g(x)}$$

are continuous on (a,b), then there exists a unique solution $y(x)$ to $(**)$, defined on an interval $I = (x_0 - h, x_0 + h)$; $h > 0$, contained in (a,b).



Rewrite the equation as:

$$y' = f(x,y) \ ; \ y(x_0) = y_0$$

where $f(x,y) = g(x) - p(x)y$. Then $\frac{\partial f}{\partial y}(x,y) = -p(x)$ so both f and $\frac{\partial f}{\partial y}$ are continuous on $(a,b) \times \mathbb{R}$.

Applying Theorem 1 to the (infinite) rectangle $R = (a,b) \times \mathbb{R}$, this result follows.