Name: ______________________________

February 10^{th}, 2016.
Math 2551; Sections L1, L2, L3.
Georgia Institute of Technology
EXAM 1

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community. By signing my name below I pledge that I have neither given nor received help on this exam.

Pledged: ______________________________

Problem	Possible Score	Earned Score
1	20	
2	16	
3	16	
4	18	
5	20	
6	10	
Total	100	

Remember that you must SHOW YOUR WORK to receive credit!

Good luck!

Angle θ ($0 \leq \theta \leq \pi$) between vectors $\mathbf{u}$ and $\mathbf{v}$:

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}; \quad \sin\theta = \frac{|\mathbf{u}\times\mathbf{v}|}{|\mathbf{u}||\mathbf{v}|}.$$

Vector Projection of $\mathbf{u}$ onto $\mathbf{v} \neq 0$:

$$\mathrm{Proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v} = |\mathbf{u}\cos\theta|\frac{\mathbf{v}}{|\mathbf{v}|}.$$

Distance from a point S to a line L going through P and parallel to $\mathbf{v}$:

$$d = \frac{|\overrightarrow{PS}\times\mathbf{v}|}{|\mathbf{v}|}$$

Length of a smooth curve C: $\mathbf{r}(t)$, traced exactly once as $a \leq t \leq b$:

$$L = \int_a^b |\mathbf{v}(t)|\,dt.$$

Arclength parameter:

$$s(t) = \int_{t_0}^t |\mathbf{v}(\tau)|\,d\tau.$$

TNB Frame:

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}; \quad \mathbf{N} = \frac{{}^{d\mathbf{T}}/_{ds}}{\kappa} = \frac{{}^{d\mathbf{T}}/_{dt}}{|{}^{d\mathbf{T}}/_{dt}|}; \quad \mathbf{B} = \mathbf{T}\times\mathbf{N}.$$

Curvature:

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}.$$

Tangential and Normal Components of Acceleration:

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N};$$

$$a_T = \frac{d^2s}{dt^2} = \frac{d}{dt}|\mathbf{v}(t)|;$$

$$a_N = \kappa\left(\frac{ds}{dt}\right)^2 = \kappa|\mathbf{v}(t)|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}.$$

Torsion:

$$\tau = -\frac{d\mathbf{B}}{ds}\cdot\mathbf{N} = -\frac{1}{|\mathbf{v}|}\frac{d\mathbf{B}}{dt}\cdot\mathbf{N}.$$

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1. [20 points] Consider the vectors:

$$\mathbf{u} = \langle 1, 0, 2 \rangle \,,$$
$$\mathbf{v} = \langle -1, 2, 1 \rangle \,.$$

a). Find $\mathbf{u} \cdot \mathbf{v}$.

b). Find the angle θ between the two vectors. Give an exact answer.

c). Find $\mathbf{u} \times \mathbf{v}$.

d). Find $\mathbf{v} \times \mathbf{u}$.

2. [16 points] Find parametric equations for the line tangent to the curve

$$\mathbf{r}(t) = \left\langle e^t,\ te^t,\ te^{t^2} \right\rangle,$$

at the point $(1, 0, 0)$.

3. [16 points] Find the length of the curve:

$$\mathbf{r}(t) = \left\langle e^t, e^t \sin t, e^t \cos t \right\rangle,$$

between the points $(1, 0, 1)$ and $(e^{2\pi}, 0, e^{2\pi})$.

4. [18 points] Consider the curve:

$$\mathbf{r}(t) = \langle 3\cos t, 3\sin t, 4t \rangle\,,$$

and its unit tangent and unit normal vectors:

$$\mathbf{T} = \left\langle -\frac{3}{5}\sin t, \frac{3}{5}\cos t, \frac{4}{5} \right\rangle,$$

$$\mathbf{N} = \langle -\cos t, -\sin t, 0 \rangle\,.$$

a). Find the unit binormal vector **B**.

b). Find the torsion τ along this curve.

5. [20 points] Find the following limits:

a). $\lim_{(x,y)\to(0,0)} \dfrac{-2e^{-4y}\sin(3x)}{-x}$

b). $\lim_{\substack{(x,y)\to(0,0)\\ x\neq y}} \dfrac{x-y+5\sqrt{x}-5\sqrt{y}}{\sqrt{x}-\sqrt{y}}$

Consider the function

$$f(x,y)=\frac{y^4-2x^2}{y^4+x^2}.$$

c). Find the limit of $f(x,y)$ as (x,y) approaches $(0,0)$ along the x-axis.

d). Find the limit of $f(x,y)$ as (x,y) approaches $(0,0)$ along the y-axis.

e). What conclusion can you draw about

$$\lim_{(x,y)\to(0,0)} \frac{y^4-2x^2}{y^4+x^2}?$$

6. [10 points] Find the point on the curve

$$\mathbf{r}(t) = \left\langle t^3, t^2 + 1, t - 1 \right\rangle$$

where the normal plane is *orthogonal* to the plane $\frac{1}{3}x - y + z = 4$. Recall that the *normal plane* is determined by $\mathbf{N}$ and $\mathbf{B}$.