

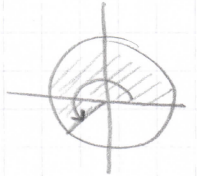
## Worksheet (2) - Solutions

- ① a). Vector      c). Scalar      e). Vector  
 b). Scalar      d). Scalar

②  $\|\vec{u}\| = \sqrt{2}$  ;  $\|\vec{v}\| = 1$ .

a).  $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos(\pi/4)$   
 $= \sqrt{2} \cdot \frac{1}{\sqrt{2}} = \boxed{1}$

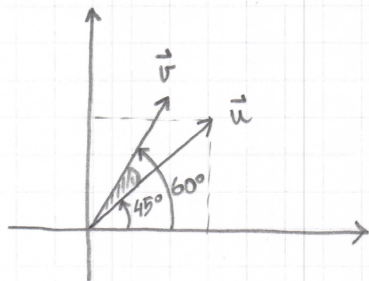
b).  $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos(3\pi/4)$   
 $= \sqrt{2} \cdot \frac{1}{\sqrt{2}} = \boxed{-1}$



③  $\|\vec{u} + \vec{v}\|^2 = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v})$   
 $= \vec{u} \cdot \vec{u} + \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v}$   
 $= \|\vec{u}\|^2 + 2\vec{u} \cdot \vec{v} + \|\vec{v}\|^2$

So if  $\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$ , then  $2\vec{u} \cdot \vec{v} = 0$ , thus  $\vec{u} \perp \vec{v}$ .

④  $\|\vec{u}\| = \|\vec{v}\| = 1$ .

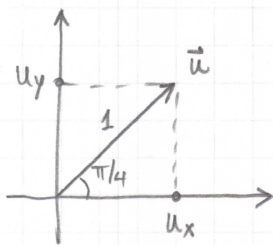


The angle between  $\vec{u}$  and  $\vec{v}$  is  $15^\circ$ , so

$$\cos(15^\circ) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} = \vec{u} \cdot \vec{v}$$

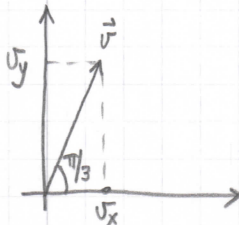
$$\left. \begin{aligned} \vec{u} &= \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle \\ \vec{v} &= \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle \end{aligned} \right\} \Rightarrow \cos(15^\circ) = \frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}$$

$$\cos(15^\circ) = \frac{1 + \sqrt{3}}{2\sqrt{2}}$$



$$\cos \frac{\pi}{4} = \frac{u_x}{1} \Rightarrow u_x = \frac{1}{\sqrt{2}}$$

$$\sin \frac{\pi}{4} = \frac{u_y}{1} \Rightarrow u_y = \frac{1}{\sqrt{2}}$$



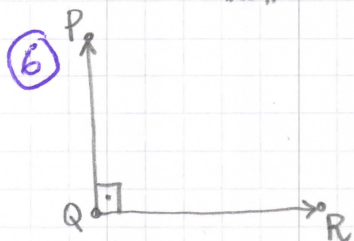
$$\cos \frac{\pi}{3} = \frac{v_x}{1} \Rightarrow v_x = \frac{1}{2}$$

$$\sin \frac{\pi}{3} = \frac{v_y}{1} \Rightarrow v_y = \frac{\sqrt{3}}{2}$$

$$\textcircled{5} \vec{b} = \langle 2, 2, 1 \rangle \quad \|\vec{b}\| = 3$$

$$\theta = \frac{\pi}{6}; \vec{a} \cdot \vec{b} = 6$$

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|} \Rightarrow \frac{\sqrt{3}}{2} = \frac{6}{\|\vec{a}\| \cdot 3} \Rightarrow \boxed{\|\vec{a}\| = \frac{4}{\sqrt{3}}}$$



$$P(a, 1, -1)$$

$$Q(0, 1, 1)$$

$$R(a, -1, 3)$$

$$\vec{QP} = \langle a, 0, -2 \rangle$$

$$\vec{QR} = \langle a, -2, 2 \rangle$$

$$\vec{QP} \cdot \vec{QR} = 0$$

$$a^2 - 4 = 0 \Rightarrow \boxed{a = \pm 2}$$

$$\textcircled{7} \text{ a.) } \vec{u} \cdot \vec{v} = \cos^2 \theta + \sin^2 \theta - 1 = 1 - 1 = 0 \Rightarrow \underline{\text{orthogonal}}$$

$$\text{ b.) } \vec{u} \cdot \vec{v} = \cos \theta \sin \theta - \sin \theta \cos \theta - 1 = -1 \Rightarrow \underline{\text{not orthogonal}}$$

$$\textcircled{8} |\vec{u} \cdot \vec{v}| = |\|\vec{u}\| \|\vec{v}\| \cos \theta| = \|\vec{u}\| \|\vec{v}\| |\cos \theta| \leq \|\vec{u}\| \|\vec{v}\|, \text{ because } |\cos \theta| \leq 1.$$

$$\textcircled{9} \|\vec{u} + \vec{v}\|^2 = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) = \vec{u} \cdot \vec{u} + \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v}$$

$$= \|\vec{u}\|^2 + 2\vec{u} \cdot \vec{v} + \|\vec{v}\|^2$$

$$\|\vec{u} - \vec{v}\|^2 = (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) = \vec{u} \cdot \vec{u} - \vec{u} \cdot \vec{v} - \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v}$$

$$= \|\vec{u}\|^2 - 2\vec{u} \cdot \vec{v} + \|\vec{v}\|^2$$

$$\Rightarrow \|\vec{u} + \vec{v}\|^2 + \|\vec{u} - \vec{v}\|^2 = 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2$$