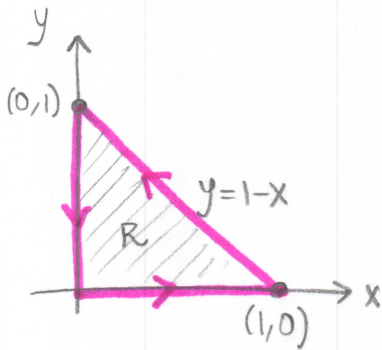


$$\textcircled{1} \oint_C x^4 dx + xy dy$$



Green's Theorem:

$$\oint_C x^4 dx + xy dy = \oint_C M dx + N dy$$

$$M = x^4$$

$$N = xy$$

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

$$= \iint_R (y - 0) dA$$

$$= \int_0^1 \int_0^{1-x} y dy dx$$

$$= \int_0^1 \frac{y^2}{2} \Big|_{y=0}^{y=1-x} dx = \frac{1}{2} \int_0^1 (1-x)^2 dx$$

$$= -\frac{1}{6} (1-x)^3 \Big|_0^1 = \boxed{\frac{1}{6}}$$

$$\textcircled{2} \oint_C (3y - e^{\sin x}) dx + (7x + \sqrt{y^4 + 1}) dy ; C: x^2 + y^2 = 9$$

$$= \oint_C M dx + N dy$$

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

$$= \iint_R (7 - 3) dA$$

$$= \iint_R 4 dA$$

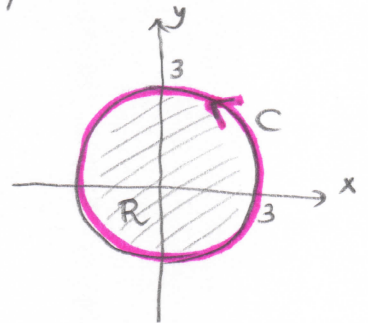
$$= 4 \text{ Area}(R)$$

$$= 4 (\pi \cdot 3^2)$$

$$= \boxed{36\pi}$$

$$M = 3y - e^{\sin x}$$

$$N = 7x + \sqrt{y^4 + 1}$$



③ Area enclosed by an ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$$C: \vec{r}(t) = \langle a \cos t, b \sin t \rangle, 0 \leq t \leq 2\pi$$

$$\text{Area} = \frac{1}{2} \oint_C x dy - y dx$$

$$x = a \cos t; dx = -a \sin t dt$$

$$y = b \sin t; dy = b \cos t dt$$

$$= \frac{1}{2} \int_0^{2\pi} (a \cos t)(b \cos t) - (b \sin t)(-a \sin t) dt$$

$$= \frac{1}{2} \int_0^{2\pi} ab (\cos^2 t + \sin^2 t) dt$$

$$= \frac{1}{2} \int_0^{2\pi} ab dt = \frac{1}{2} ab t \Big|_0^{2\pi} = \boxed{\pi ab}$$

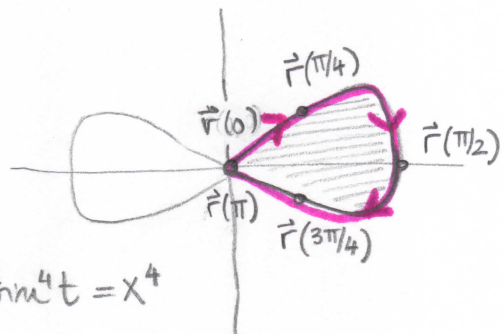
④ Lemniscate: $x^4 = x^2 - y^2$

$$(a). \vec{r}(t) = \langle \sin t, \sin t \cos t \rangle; 0 \leq t \leq \pi$$

$$x = \sin t; y = \sin t \cos t$$

$$x^2 - y^2 = \sin^2 t - \sin^2 t \cos^2 t = \sin^2 t (1 - \cos^2 t) = \sin^2 t = x^4$$

The restriction $0 \leq t \leq \pi$ means $\sin t \geq 0$
 so $x = \sin t \geq 0$ (the right lobe).



$$(b). \text{Area} = \frac{1}{2} \oint_C x dy - y dx = \frac{1}{2} \int_0^{\pi} ((\sin t)(2 \cos^2 t - 1) - \sin t \cos t \cdot \cos t) dt$$

$$x = \sin t; dx = \cos t dt$$

$$= \frac{1}{2} \int_0^{\pi} (2 \sin t \cos^2 t - \sin t - \sin t \cos^2 t) dt$$

$$y = \sin t \cos t; dy = \cos(2t) dt$$

$$= \frac{1}{2} \sin(2t); dy = (\cos^2 t - \sin^2 t) dt$$

$$= (2 \cos^2 t - 1) dt$$

$$= \frac{1}{2} \int_0^{\pi} (\sin t \cos^2 t - \sin t) dt$$

$$= \frac{1}{2} \left(-\frac{1}{3} \cos^3 t + \cos t \right) \Big|_0^{\pi}$$

$$= \frac{1}{2} \left(\frac{1}{3} - 1 + \frac{1}{3} - 1 \right) = \frac{1}{2} \left(\frac{2}{3} - 2 \right) = -\frac{2}{3} ?$$

What happened? The parametrization has the curve negatively oriented:

$$\vec{r}(0) = \langle 0, 0 \rangle; \vec{r}\left(\frac{\pi}{4}\right) = \left\langle \frac{\sqrt{2}}{2}, \frac{1}{2} \right\rangle; \vec{r}\left(\frac{\pi}{2}\right) = \langle 1, 0 \rangle; \vec{r}\left(\frac{3\pi}{4}\right) = \left\langle \frac{\sqrt{2}}{2}, -\frac{1}{2} \right\rangle; \vec{r}(\pi) = \langle 0, 0 \rangle$$

So really:

$$\text{Area} = -\left(\frac{1}{2} \oint_C x dy - y dx \right) \Rightarrow \text{Area} = \boxed{\frac{2}{3}}$$