Worksheet 15 Solutions
Part I - Line Integrals

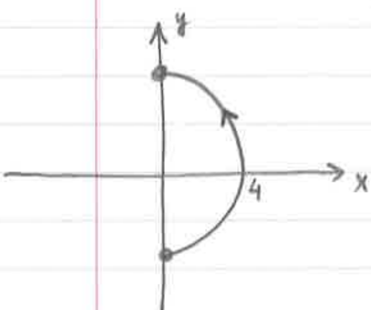
4) $f(x,y,z) = xyz$

$C: \vec{r}(t) = \cos(t)\,\vec{i} + \sin(t)\,\vec{j} + 3t\,\vec{k}, \quad 0 \le t \le 4\pi.$

$\vec{v}(t) = \langle -\sin t, \cos t, 3 \rangle \;; \quad |\vec{v}(t)| = \sqrt{\sin^2 t + \cos^2 t + 9} = \sqrt{10}$

$$\int_C f\,ds = \int_0^{4\pi} \cos(t)\sin(t)\cdot 3t \cdot \sqrt{10}\,dt$$
$$= 3\sqrt{10}\int_0^{4\pi} t\sin(t)\cos(t)\,dt$$
$$= 3\sqrt{10}\int_0^{4\pi} t\cdot\frac{1}{2}\sin(2t)\,dt$$
$$= \frac{3\sqrt{10}}{2}\int_0^{4\pi} t\cdot(\cos(2t))'\,dt\left(\frac{-1}{2}\right)$$
$$= \frac{-3\sqrt{10}}{4}\left(t\cos(2t)\Big|_0^{4\pi} - \int_0^{4\pi}\cos(2t)\,dt\right)$$
$$= \frac{-3\sqrt{10}}{4}\left(4\pi - 0 - \underbrace{\frac{1}{2}\sin(2t)\Big|_0^{4\pi}}_{0}\right) = \boxed{-3\sqrt{10}\,\pi}$$

1) $\int_C xy^4\,ds$, C: right half of circle $x^2+y^2=16$

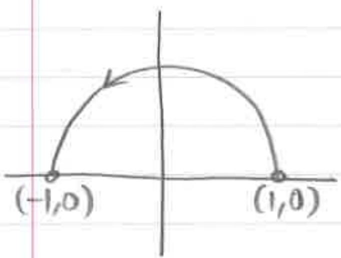


$x = 4\cos t$
$y = 4\sin t$
$-\pi/2 \le t \le \pi/2$

$\vec{r}(t) = \langle 4\cos t, 4\sin t \rangle$
$\vec{v}(t) = \langle -4\sin t, 4\cos t \rangle$
$|\vec{v}(t)| = \sqrt{16\sin^2 t + 16\cos^2 t} = \sqrt{16} = 4.$

$$\int_C xy^4\,ds = \int_{-\pi/2}^{\pi/2} 4\cos t\,(4\sin t)^4\cdot 4\,dt$$
$$= 4^6\left(\frac{\sin^5 t}{5}\right)\Big|_{-\pi/2}^{\pi/2} = 4^6\cdot\left(\frac{1}{5} - \left(-\frac{1}{5}\right)\right) = \frac{2\cdot 4^6}{5}$$
$$= \boxed{\frac{8192}{5}}$$

(2) $\int_C (2+x^2y)\,ds$, C: upper half of unit circle $x^2+y^2=1$.



$\vec{r}(t) = \langle \cos t, \sin t \rangle$ $\quad$ $\vec{v}(t) = \langle -\sin t, \cos t \rangle$

$0 \le t \le \pi$ $\quad$ $|\vec{v}(t)| = 1$

$$\int_C (2+x^2y)\,ds = \int_0^{\pi} \left(2 + \cos^2 t \sin t\right) dt$$

$$= 2t\Big|_0^{\pi} - \frac{\cos^3 t}{3}\Big|_0^{\pi}$$

$$= 2\pi - \left(\frac{(-1)^3}{3} - \frac{1}{3} \right) = \boxed{2\pi + 2/3}$$

(3) $\int_C \frac{x^2}{y^{4/3}}\,ds$; C: $\vec{r}(t) = \langle t^2, t^3 \rangle$, $-3 \le t \le 1$.

$\vec{v}(t) = \langle 2t, 3t^2 \rangle$; $|\vec{v}(t)| = \sqrt{4t^2+9t^4} = \sqrt{t^2(4+9t^2)}$

$= |t|\sqrt{4+9t^2}$

$$\int_C \frac{x^2}{y^{4/3}}\,ds = \int_{-3}^{1} \underbrace{\frac{(t^2)^2}{(t^3)^{4/3}}}_{=1} |t|\sqrt{4+9t^2}\,dt$$

$$= \int_{-3}^{0} (-t)\sqrt{4+9t^2}\,dt + \int_0^1 t\sqrt{4+9t^2}\,dt$$

$$= -\frac{1}{9}\cdot\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}(4+9t^2)^{3/2}\Big|_{-3}^{0} + \frac{1}{9}\cdot\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}(4+9t^2)^{3/2}\Big|_0^1$$

$$= \frac{-1}{27}\left(4^{3/2} - 85^{3/2}\right) + \frac{1}{27}\left(13^{3/2} - 4^{3/2}\right)$$

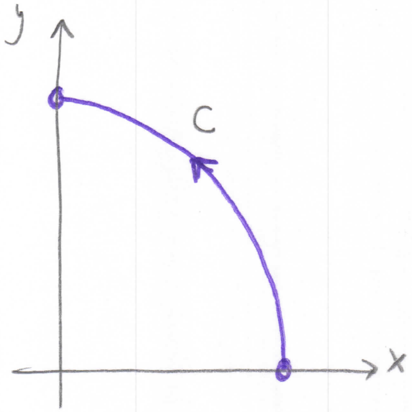
$$= \frac{1}{27}\left(85\sqrt{85} - 8 + 13\sqrt{13} - 8\right)$$

$$= \boxed{\frac{1}{27}\left(85\sqrt{85} + 13\sqrt{13} - 16\right)}$$

Worksheet 15 Solutions
Part II - Flow, Circulation and Flux

① $\vec{F}(x,y) = \langle x^2, -xy \rangle$

C: $\vec{r}(t) = \langle \cos t, \sin t \rangle$, $0 \le t \le \pi/2$



$x = \cos t$; $dx = -\sin t\, dt$

$y = \sin t$; $dy = \cos t\, dt$

$$W = \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_C M\,dx + N\,dy = \int_C x^2 dx - xy\,dy$$

$$= \int_0^{\pi/2} \left((\cos^2 t)(-\sin t) - (\cos t \sin t) \cos t \right) dt$$

$$= \int_0^{\pi/2} -2\cos^2 t \sin t \, dt$$

$$= \frac{2}{3} \cos^3 t \Big|_0^{\pi/2} = \frac{2}{3}(0-1) = -2/3$$

② $\vec{F}(x,y,z) = \langle xy, yz, zx \rangle$

C: $\vec{r}(t) = \langle t, t^2, t^3 \rangle$, $0 \le t \le 1$

$x = t$; $dx = dt$

$y = t^2$; $dy = 2t\,dt$

$z = t^3$; $dz = 3t^2 dt$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C xy\,dx + yz\,dy + zx\,dz$$

$$= \int_0^1 \left(t^3 + t^5 \cdot 2t + t^4 \cdot 3t^2 \right) dt$$

$$= \int_0^1 \left(t^3 + 5t^6 \right) dt$$

$$= \left(\frac{t^4}{4} + \frac{5t^7}{7} \right) \Big|_0^1 = \frac{1}{4} + \frac{5}{7} = \frac{27}{28}$$

③ $\vec{F}(x,y) = \langle x^2y^3, -y\sqrt{x} \rangle$

C: $\vec{r}(t) = \langle t^2, -t^3 \rangle$, $0 \le t \le 1$

$x = t^2$; $dx = 2t\,dt$

$y = -t^3$; $dy = -3t^2dt$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C x^2y^3dx - y\sqrt{x}\,dy$$

$\sqrt{t^2} = t$ because $0 \le t \le 1$!

$$= \int_0^1 \left[t^4(-t^9)(2t) - (-t^3)\,t\,(-3t^2) \right] dt$$

$$= \int_0^1 \left(-2t^{14} - 3t^6 \right) dt$$

$$= \left(-\frac{2}{15}t^{15} - \frac{3}{7}t^7 \right)\Big|_0^1 = -\frac{2}{15} - \frac{3}{7} = -\frac{59}{105}$$

④ $\vec{F}(x,y,z) = \langle \sin x, \cos y, xz \rangle$

C: $\vec{r}(t) = \langle t^3, -t^2, t \rangle$; $0 \le t \le 1$

$x = t^3$; $dx = 3t^2dt$

$y = -t^2$; $dy = -2t\,dt$

$z = t$; $dz = dt$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \sin x\,dx + \cos y\,dy + xz\,dz$$

$$= \int_0^1 \left(\sin(t^3)(3t^2) + \cos(-t^2)(-2t) + t^3 \cdot t \cdot 1 \right) dt$$

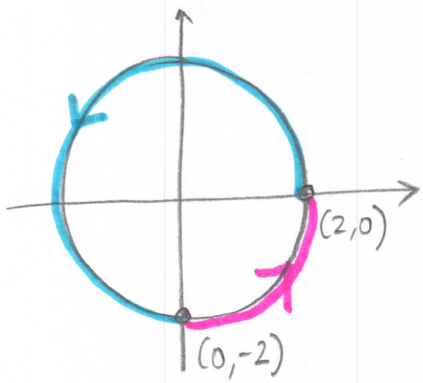
$$= \left(-\cos(t^3) + \sin(-t^2) + \frac{1}{5}t^5 \right)\Big|_0^1$$

$$= -\cos(1) + \sin(-1) + \frac{1}{5} - (\underbrace{-\cos(0)}_{-1} + \sin(0) + 0)$$

$$= \boxed{\frac{6}{5} - \cos(1) - \sin(1)}$$

5 $\vec{F}(x,y) = \langle x-y, xy \rangle$

C_1: arc of circle $x^2+y^2=4$ traversed counterclockwise from $(2,0)$ to $(0,-2)$



$C_1: \vec{r}(t) = \langle 2\cos t, 2\sin t \rangle, \; 0 \le t \le \frac{3\pi}{2}$.

$x = 2\cos t$; $dx = -2\sin t\, dt$

$y = 2\sin t$; $dy = 2\cos t\, dt$

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_1} (x-y)\,dx + (xy)\,dy$$

$$= \int_0^{3\pi/2} (2\cos t - 2\sin t)(-2\sin t) + 4\sin t\cos t\,(2\cos t)\,dt$$

$$= \int_0^{3\pi/2} \left(-4\sin t\cos t + 4\sin^2 t + 8\sin t\cos^2 t\right)dt$$

$$= \left(2\cos^2 t + 2t - \sin(2t) - \frac{8}{3}\cos^3 t\right)\Big|_0^{3\pi/2}$$

$$= (3\pi) - \left(2 - \frac{8}{3}\right) = \boxed{3\pi + \frac{2}{3}}$$

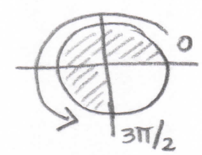
$$\int \sin^2 t\, dt = \int \frac{1-\cos(2t)}{2}\,dt = \frac{t}{2} - \frac{1}{4}\sin(2t)$$

C_2: arc of the circle $x^2+y^2=4$ traversed counterclockwise from $(0,-2)$ to $(2,0)$:

$C_2: \vec{r}(t) = \langle 2\cos t, 2\sin t \rangle$; $\boxed{-\frac{\pi}{2} \le t \le 0}$ or $\boxed{\frac{3\pi}{2} \le t \le 2\pi}$

Careful here and don't take $\int_{3\pi/2}^{0} \dots dt$

this is really $-\int_0^{3\pi/2} \dots dt$



$$\int_{C_2} \vec{F}\cdot d\vec{r} = \int_{-\pi/2}^{0} (x-y)\,dx + (xy)\,dy$$

$= \dots$ same integral

$$= \left(2\cos^2 t + 2t - \sin(2t) - \frac{8}{3}\cos^3 t\right)\Big|_{-\pi/2}^{0} \text{ or } \Big|_{3\pi/2}^{2\pi}$$

doesn't matter which one

$$= \left(2 - \frac{8}{3}\right) - (-\pi) = \boxed{\pi - \frac{2}{3}}$$

(c). Circulation of $\vec{F}$ along circle $x^2+y^2=4$ (C):

$$\oint_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} = 3\pi + \frac{2}{3} + \pi - \frac{2}{3} = \boxed{4\pi}$$

(d). Flux of $\vec{F}$ about C?

$C: \vec{r}(t) = \langle 2\cos t, 2\sin t\rangle, 0 \le t \le 2\pi$

$$\int_C \vec{F}\cdot\vec{n}\,ds = \int_C M\,dy - N\,dx$$

$$= \int_0^{2\pi} (2\cos t - 2\sin t)(2\cos t) - (4\sin t\cos t)(-2\sin t)\,dt$$

$$= \int_0^{2\pi} \left(4\cos^2 t - 4\sin t\cos t + 8\sin^2 t\cos t\right) dt$$

$$= \left(2t + \sin(2t) + 2\cos^2 t + \frac{8}{3}\sin^3 t\right)\Big|_0^{2\pi}$$

$$= \boxed{4\pi}$$

$$\int \cos^2 t\,dt = \int \frac{1+\cos(2t)}{2}\,dt = \frac{t}{2} + \frac{1}{4}\sin(2t)$$

---

6. $\vec{F}(x,y) = \langle e^{x-1}, xy\rangle$

$C: \vec{r}(t) = \langle t^2, t^3\rangle; 0 \le t \le 1$

$x = t^2$; $dx = 2t\,dt$

$y = t^3$; $dy = 3t^2\,dt$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C e^{x-1}dx + xy\,dy$$

$$= \int_0^1 \left(e^{t^2-1}\cdot 2t + t^5\cdot 3t^2\right) dt$$

$$= \int_0^1 \left(e^{t^2-1}\cdot 2t + 3t^7\right) dt$$

$$= \left(e^{t^2-1} + \frac{3}{8}t^8\right)\Big|_0^1$$

$$= 1 + \frac{3}{8} - \frac{1}{e} = \boxed{\frac{11}{8} - \frac{1}{e}}$$

7 $\vec{F}(x,y) = \langle x,\ y+2 \rangle$

$C: \vec{r}(t) = \langle t - \sin t,\ 1-\cos t \rangle\ ;\ 0 \le t \le 2\pi$

$x = t - \sin t\ ;\ dx = (1-\cos t)\,dt$

$y = 1 - \cos t\ ;\ dy = \sin t\ dt$

$$\text{Work} = \int_C \vec{F}\cdot d\vec{r}$$

$$= \int_C M\,dx + N\,dy$$

$$= \int_0^{2\pi} \Big((t-\sin t)(1-\cos t) + (3-\cos t)(\sin t) \Big)\,dt$$

$$= \int_0^{2\pi} \left(t - t\cos t - \sin t + \cancel{\sin t\cos t} + 3\sin t - \cancel{\cos t\sin t} \right) dt$$

$$= \int_0^{2\pi} (t - t\cos t + 2\sin t)\,dt$$

$$= \left(\frac{t^2}{2} - t\sin t - \cos t - 2\cos t \right)\Big|_0^{2\pi}$$

$$= \frac{(2\pi)^2}{2} = \boxed{2\pi^2}$$

$\int t\cos t\,dt$

$= \int t(\sin t)'\,dt$

$= t\sin t - \int \sin t\,dt$

$= t\sin t + \cos t$

8 $\vec{F}(x,y) = \left\langle \dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2} \right\rangle$

$C: \vec{r}(t) = \langle \cos t, \sin t \rangle, \quad 0 \le t \le 2\pi$

Flux: $\oint_C \vec{F}\cdot\vec{n}\, ds = \oint_C M\,dy - N\,dx$

$= \int_0^{2\pi} \left[(-\sin t)(\cos t) - (\cos t)(-\sin t)\right] dt$

$= \int_0^{2\pi} 0\, dt = \boxed{0}$

$x = \cos t \;;\; dx = -\sin t\, dt$

$y = \sin t \;;\; dy = \cos t\, dt$

$M = \dfrac{-y}{x^2+y^2} = \dfrac{-\sin t}{\sin^2 t + \cos^2 t} = -\sin t$

$N = \dfrac{x}{x^2+y^2} = \dfrac{\cos t}{\cos^2 t + \sin^2 t} = \cos t$

Circulation: $\oint_C \vec{F}\cdot d\vec{r}$

$= \oint_C M\,dx + N\,dy$

$= \int_0^{2\pi} \Big((-\sin t)(-\sin t) + (\cos t)(\cos t)\Big) dt$

$= \int_0^{2\pi} 1\, dt = \boxed{2\pi}$