Worksheet 11 - Solutions

1 a). $\int_1^3\int_0^2 x^3y\,dy\,dx = \int_1^3 x^3\frac{y^2}{2}\Big|_{y=0}^{y=2}dx$

$= \int_1^3 2x^3dx = 2\frac{x^4}{4}\Big|_1^3 = \frac{81}{2}-\frac{1}{2} = \boxed{40}$

b). $\int_0^2\int_1^3 x^3y\,dy\,dx = \int_0^2 x^3\frac{y^2}{2}\Big|_{y=1}^{y=3}dx$

$= \int_0^2\left(\frac{9}{2}-\frac{1}{2}\right)x^3dx = \int_0^2 4x^3dx = x^4\Big|_0^2 = \boxed{16}$

c). $\int_0^1\int_0^2 (x+4y^3)\,dx\,dy = \int_0^1\left(\frac{x^2}{2}+4y^3x\right)\Big|_{x=0}^{x=2}dy$

$= \int_0^1(2+8y^3)dy = (2y+2y^4)\Big|_0^1 = \boxed{4}$

d). $\int_0^1\int_2^3 \sqrt{x+4y}\,dx\,dy = \int_0^1 \frac{2}{3}(x+4y)^{3/2}\Big|_{x=2}^{x=3}dy$

$= \frac{2}{3}\int_0^1\left[(3+4y)^{3/2}-(2+4y)^{3/2}\right]dy$

$= \frac{2}{3}\left(\frac{2}{5}\frac{1}{4}(3+4y)^{5/2}-\frac{2}{5}\frac{1}{4}(2+4y)^{5/2}\right)\Big|_0^1$

$= \boxed{\frac{1}{15}\left(7^{5/2}-6^{5/2}-3^{5/2}+2^{5/2}\right)} \approx \boxed{2.102}$

e). $\int_1^2\int_0^4 \frac{dy\,dx}{x+y} = \int_1^2 \ln(x+y)\Big|_{y=0}^{y=4}dx$

$= \int_1^2\left[\ln(x+4)-\ln(x)\right]dx$

$= \left[(x+4)\ln(x+4)-(x+4)-x\ln(x)+x\right]\Big|_{x=1}^{2}$

$= \left((x+4)\ln(x+4)-x\ln(x)-4\right)\Big|_1^2$

$= 6\ln(6)-2\ln(2)-\cancel{4}-5\ln(5)+\cancel{4}$

$= \boxed{6\ln(6)-2\ln(2)-5\ln(5)} \approx \boxed{1.31.}$

$\boxed{\int \ln(x)dx = x\ln(x)-x+C}$ $\longrightarrow$ By parts:

$\int\ln(x)dx = \int 1\cdot\ln(x)dx = \int (x)'\ln(x)dx$

$= x\ln(x)-\int x(\ln(x))'dx$

$= x\ln(x)-\int x\cdot\frac{1}{x}dx = x\ln(x)-\int 1dx = x\ln(x)-x$

f). $\int_1^2 \int_1^2 \ln(xy)\,dy\,dx = \int_1^2 \left(\frac{1}{x}\left(xy\ln(xy) - xy\right)\right)\Big|_{y=1}^{y=2} dx$

$$= \int_1^2 \left(y\ln(xy) - y\right)\Big|_{y=1}^{y=2} dx$$

$$= \int_1^2 \left(2\ln(2x) - 2 - \ln(x) + 1\right) dx$$

$$= \int_1^2 \left(2\ln(2x) - \ln(x) - 1\right) dx$$

$$= \left(2x\ln(2x) - 2x - x\ln(x) + x - x\right)\Big|_1^2$$

$$= \left(2x\ln(2x) - x\ln(x) - 2x\right)\Big|_1^2$$

$$= 4\ln(4) - 2\ln(2) - 4 - 2\ln(2) + 2$$

$$= \boxed{4\ln(4) - 4\ln(2) - 2} = \boxed{4\ln(2) - 2}$$

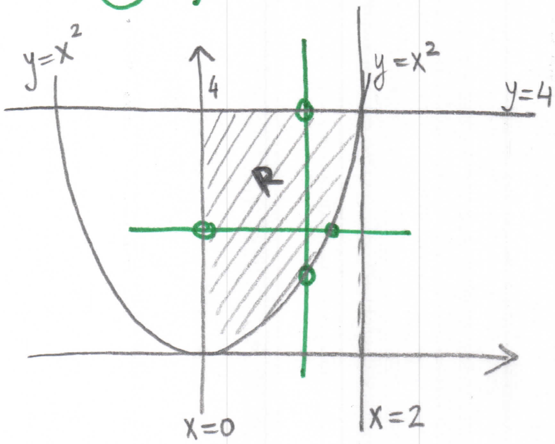
2 $f(x,y) = mxy^2$; $\iint_R f(x,y)\,dA = 1$; $R = [0,1]\times[0,2]$; $m = ?$

$$\int_0^2 \int_0^1 mxy^2\,dx\,dy = m\int_0^2 \int_0^1 xy^2\,dx\,dy = m\int_0^2 \frac{x^2}{2}y^2\Big|_{x=0}^{x=1} dy$$

$$= m\int_0^2 \frac{1}{2}y^2\,dy = m\,\frac{y^3}{6}\Big|_0^2 = m\,\frac{4}{3}$$

$\Rightarrow \frac{4}{3}m = 1 \Rightarrow \boxed{m = 3/4}$

③ a). $0 \le x \le 2$, $x^2 \le y \le 4$



$\iint_R x^3 \, dA$

Horizontal Cross-Sections :

$$\int_0^4 \int_0^{\sqrt{y}} x^3 \, dx \, dy = \int_0^4 \frac{x^4}{4} \Big|_{x=0}^{x=\sqrt{y}} dy$$

$$= \int_0^4 \frac{y^2}{4} dy = \frac{y^3}{12}\Big|_0^4 = \frac{64}{12} = \boxed{\frac{16}{3}}$$

Vertical Cross-Sections :

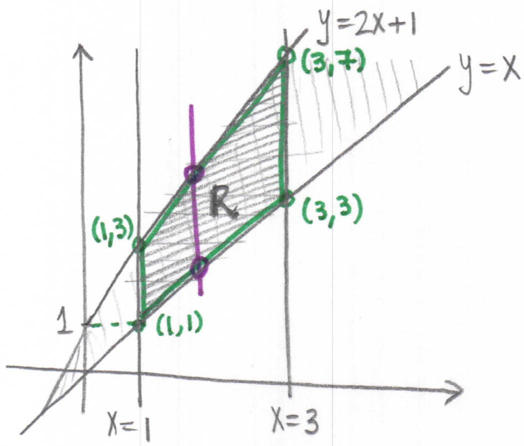
$$\int_0^2 \int_{x^2}^4 x^3 \, dy \, dx = \int_0^2 \left(x^3 y \Big|_{y=x^2}^{y=4} \right) dx$$

$$= \int_0^2 \left(4x^3 - x^5\right) dx$$

$$= \left(x^4 - \frac{x^6}{6}\right)\Big|_0^2 = 16 - \frac{64}{6}$$

$$= 16 - \frac{32}{3} = \boxed{\frac{16}{3}}$$

b). $1 \le x \le 3$; $x \le y \le 2x+1$



$f(x,y) = x^2 y$

Remark : If we do horizontal cross-sections, we would have to split into two integrals because the bounds for x change between $y \in [1,3]$ and $y \in [3,7]$.

Vertical cross-sections :

$$\int_1^3 \int_x^{2x+1} x^2 y \, dy \, dx$$

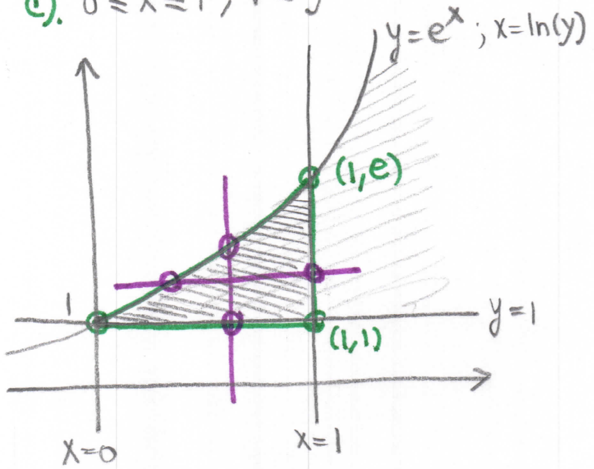
$$= \int_1^3 x^2 \frac{y^2}{2} \Big|_{y=x}^{y=2x+1} dx$$

$$= \int_1^3 x^2 \left(\frac{(2x+1)^2}{2} - \frac{x^2}{2} \right) dx$$

$$= \frac{1}{2} \int_1^3 x^2 (3x^2 + 4x + 1) \, dx = \frac{1}{2} \int_1^3 (3x^4 + 4x^3 + x^2) \, dx$$

$$= \frac{1}{2} \left(\frac{3x^5}{5} + x^4 + \frac{x^3}{3} \right)\Big|_1^3 = \ldots = \frac{1754}{15}.$$

c). $0 \le x \le 1 ;\ 1 \le y \le e^x$



$f(x,y) = 1. \rightsquigarrow \iint_R 1\,dA = $ area of region

Vertical Cross-Sections :

$$\int_0^1 \int_1^{e^x} 1\,dy\,dx$$

$$= \int_0^1 y \Big|_{y=1}^{y=e^x} dx$$

$$= \int_0^1 (e^x - 1)\,dx = (e^x - x)\Big|_0^1 = e-1-1$$

$$= \boxed{e-2}$$

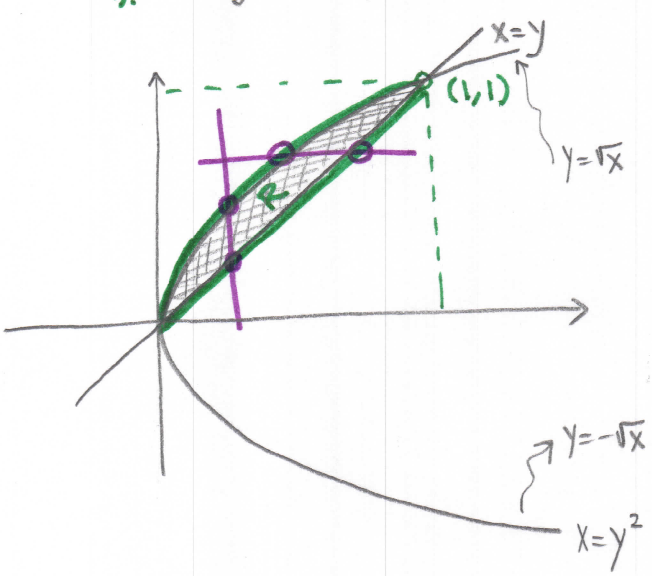
Horizontal Cross-Sections :

$$\int_1^e \int_{\ln(y)}^1 1\,dx\,dy = \int_1^e x \Big|_{x=\ln(y)}^{x=1} dy$$

$$= \int_1^e (1-\ln(y))\,dy = (y - y\ln(y) + y)\Big|_1^e$$

$$= 2e - e\underbrace{\ln(e)}_{1} - 2 = \boxed{e-2}$$

d). $0 \le y \le 1,\ y^2 \le x \le y$



$f(x,y) = 2xy$

Horizontal Cross-Sections :

$$\int_0^1 \int_{y^2}^{y} 2xy\,dx\,dy$$

$$= \int_0^1 x^2 y \Big|_{x=y^2}^{x=y} dy$$

$$= \int_0^1 (y^3 - y^5)\,dy = \left(\frac{y^4}{4} - \frac{y^6}{6}\right)\Big|_0^1$$

$$= \frac{1}{4} - \frac{1}{6} = \boxed{\frac{1}{12}}$$

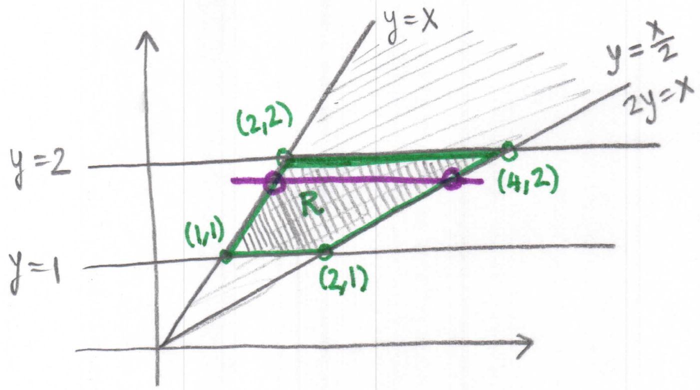
Vertical Cross-Sections :

$$\int_0^1 \int_x^{\sqrt{x}} 2xy\,dy\,dx$$

$$= \int_0^1 xy^2 \Big|_{y=x}^{y=\sqrt{x}} dx$$

$$= \int_0^1 (x^2 - x^3)\,dx = \left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1 = \frac{1}{3} - \frac{1}{4} = \boxed{\frac{1}{12}}$$

e). $y \le x \le 2y \; ; \; 1 \le y \le 2$



$f(x,y) = \dfrac{\sin y}{y}$

Can't integrate in y, so take horizontal cross-sections

$$\int_1^2 \int_y^{2y} \frac{\sin y}{y} \, dx \, dy$$

$$= \int_1^2 \left(\frac{\sin y}{y} x \Big|_{x=y}^{x=2y} \right) dy$$

$$= \int_1^2 \left(\frac{\sin y}{y} (2y - y) \right) dy$$

$$= \int_1^2 \left(\frac{\sin y}{y} \cdot y \right) dy = \int_1^2 \sin y \, dy$$

$$= -\cos(y) \Big|_1^2 = \boxed{\cos(1) - \cos(2)}$$

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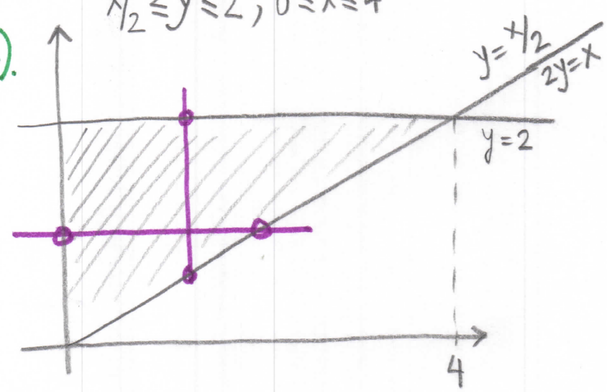
f). $x/2 \le y \le 2 \; ; \; 0 \le x \le 4$



$f(x,y) = e^{y^2}$

Can't integrate in y

$$\int_0^2 \int_0^{2y} e^{y^2} dx \, dy$$

$$= \int_0^2 x e^{y^2} \Big|_{x=0}^{x=2y} dy$$

$$= \int_0^2 2y e^{y^2} dy = e^{y^2} \Big|_0^2 = \boxed{e^4 - 1}$$