Name: Solutions

March 23^{rd}, 2015.
Math 2401; Sections K1, K2, K3.
Georgia Institute of Technology
Exam 3

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community. By signing my name below I pledge that I have neither given nor received help on this exam.

Pledged: ____________________

Problem	Possible Score	Earned Score
1	20	
2	20	
3	20	
4	18	
5	16	
6	6	
Total	100	

Remember that you must SHOW YOUR WORK to receive credit!

Good luck!

2. [20 points] Find:

$$\int_1^2 \int_1^{\sqrt{z}} \int_{\ln(x)}^{\ln(3x)} e^{x^2+y+z}\, dy\, dx\, dz.$$

$$\int_1^2 \int_1^{\sqrt{z}} \int_{\ln(x)}^{\ln(3x)} e^{x^2} \cdot e^{y} \cdot e^{z}\, dy\, dx\, dz \qquad (2 \text{ pts.})$$

$$= \int_1^2 \int_1^{\sqrt{z}} e^{x^2} \cdot e^{z} \cdot e^{y} \Big|_{y=\ln(x)}^{y=\ln(3x)} dx\, dz \qquad (4 \text{ pts.})$$

$$= \int_1^2 \int_1^{\sqrt{z}} e^{x^2} \cdot e^{z} (3x - x)\, dx\, dz$$

$$= \int_1^2 \int_1^{\sqrt{z}} e^{x^2} \cdot e^{z} (2x)\, dx\, dz \qquad (2 \text{ pts.})$$

$$= \int_1^2 e^{z} \cdot e^{x^2} \Big|_{x=1}^{x=\sqrt{z}} dz \qquad (5 \text{ pts.})$$

$$= \int_1^2 e^{z} \cdot \left(e^{z} - e\right) dz \qquad (2 \text{ pts.})$$

$$= \int_1^2 \left(e^{2z} - e^{z+1}\right) dz$$

$$= \left(\frac{1}{2} e^{2z} - e^{z+1}\right)\Big|_1^2 \qquad (3 \text{ pts.})$$

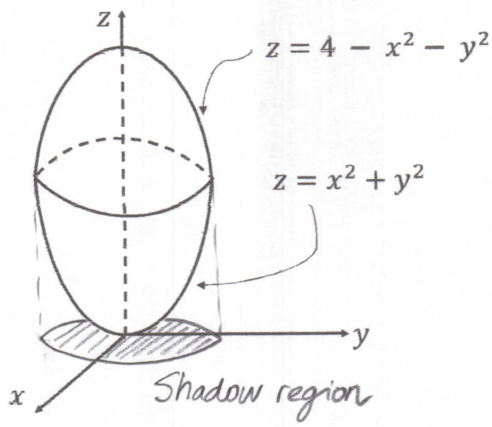
$$= \frac{1}{2} e^{4} - \frac{1}{2} e^{2} - e^{3} + e^{2}$$

$$= \boxed{\frac{1}{2} e^{4} - e^{3} + \frac{1}{2} e^{2}} \qquad (2 \text{ pts.})$$

5. [16 points] Using *cylindrical coordinates*, set up the triple integral to compute the volume of the solid enclosed by the two paraboloids:

$$z = 4 - x^2 - y^2;$$
$$z = x^2 + y^2,$$

pictured below. You do not have to compute the value of the integral.



$z = 4 - r^2$ (3 pts.)

$z = r^2$ (3 pts.)

Circle of intersection: $\begin{cases} z = 4 - r^2 \\ z = r^2 \end{cases}$

$$4 - r^2 = r^2$$
$$4 = 2r^2$$
$$2 = r^2$$
$$r = \sqrt{2}$$ (3 pts.)

$$V = \underbrace{\int_0^{2\pi}}_{\text{1pt.}} \underbrace{\int_0^{\sqrt{2}}}_{\text{2pts.}} \underbrace{\int_{r^2}^{4-r^2}}_{\text{2pts.}} \underbrace{r}_{\text{1pt.}} \, \underbrace{dz\, dr\, d\theta}_{\text{1pt.}}$$

Name: Solutions

April 15^{th}, 2015.
Math 2401; Sections K1, K2, K3.
Georgia Institute of Technology
Exam 4

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community. By signing my name below I pledge that I have neither given nor received help on this exam.

Pledged: ______________________

Problem	Possible Score	Earned Score
0	10	10
1	18	
2	18	
3	18	
4	18	
5	18	
Total	100	

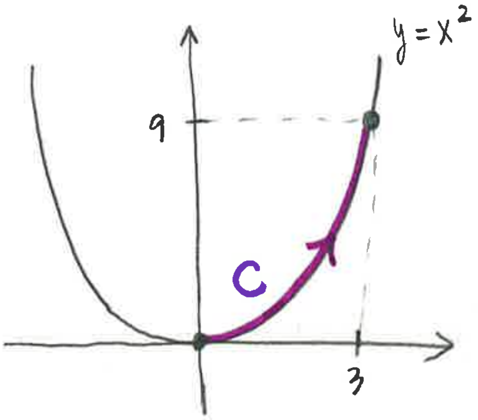
Remember that you must SHOW YOUR WORK to receive credit!

Good luck!

1. [18 points] Find the line integral:

$$\int_C 3x\,ds,$$

where C is the portion of the parabola $y = x^2$ from $(0,0)$ to $(3,9)$.



Parametrize C: (4pts.)

$t = x$ 1

$\vec{r}(t) = \langle t, t^2 \rangle$; 2

$0 \le t \le 3$ 1

Velocity: (4 pts.)

$\vec{v}(t) = \langle 1, 2t \rangle$; 2

Speed:

$|\vec{v}(t)| = \sqrt{1+4t^2}$ 2

(2pts.) Evaluate f on the curve: $f(x,y) = 3x$ 1/2

$f(\vec{r}(t)) = 3t$; 3/2

(8pts.) Compute line integral:

$\int_C 3x\,ds = \int_0^3 f(\vec{r}(t))\,|\vec{v}(t)|\,dt$ 2

$= \int_0^3 3t\sqrt{1+4t^2}\,dt$

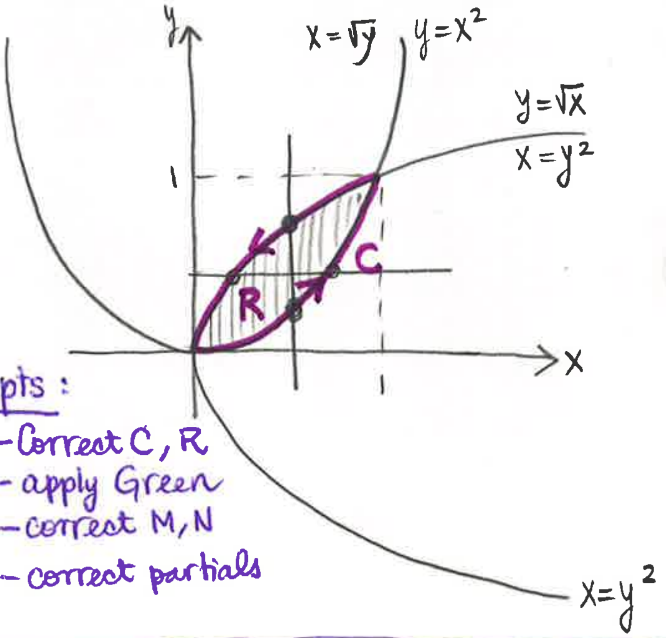
$= \cancel{3} \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{2}{\cancel{3}} (1+4t^2)^{3/2} \Big|_0^3$ 4

$= \boxed{\frac{1}{4}\left(37^{3/2} - 1\right)}$ 2

2. [18 points] Find:

$$\oint_C \left(y + e^{\sqrt{x}}\right)\, dx + \left(2x + \cos y^2\right)\, dy,$$

where C is the positively oriented boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$.



9 pts:

(3) - Correct C, R

(2) - apply Green

(2) - correct M, N

(2) - correct partials

$$\oint_C (y+e^{\sqrt{x}})\,dx + (2x+\cos y^2)\,dy$$

$$= \oint_C M\,dx + N\,dy$$

$$\boxed{\begin{aligned} M &= y+e^{\sqrt{x}} \\ N &= 2x+\cos y^2 \end{aligned}}$$

$$\underset{\text{(Green)}}{=} \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

$$= \iint_R (2-1)\,dA$$

$$= \text{Area}(R)$$

9 pts: Vertical Cross-Sections: (or) Horizontal Cross-Sections:

Vertical Cross-Sections		Horizontal Cross-Sections
$= \int_0^1 \int_{x^2}^{\sqrt{x}} 1\,dy\,dx$	3	$= \int_0^1 \int_{y^2}^{\sqrt{y}} 1\,dx\,dy$
$= \int_0^1 y \Big\vert_{y=x^2}^{y=\sqrt{x}}\,dx$	2	$= \int_0^1 x \Big\vert_{x=y^2}^{x=\sqrt{y}}\,dy$
$= \int_0^1 (\sqrt{x}-x^2)\,dx$	1	$= \int_0^1 (\sqrt{y}-y^2)\,dy$
$= \left(\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\right)\Big\vert_0^1$	2	$= \left(\frac{2}{3}y^{3/2} - \frac{1}{3}y^3\right)\Big\vert_0^1$
$= \frac{2}{3} - \frac{1}{3}$		$= \frac{2}{3} - \frac{1}{3}$
$= \boxed{\frac{1}{3}}$	1	$= \boxed{\frac{1}{3}}$

3. [18 points] Consider the conservative field:

$$\mathbf{F}(x,y,z) = (yz)\mathbf{i} + (xz - 2y\ln(z))\,\mathbf{j} + \left(xy - \frac{y^2}{z}\right)\mathbf{k}.$$

a). [12 points] Find a potential function for this field.
b). [6 points] Find $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the curve:

$$\mathbf{r}(t) = \langle t, t^2, e^t\rangle\,,\ \ 0 \le t \le 1.$$

a). $\vec{F} = \nabla f(x,y,z)$.

1 $\frac{\partial f}{\partial x} = yz \Rightarrow f = xyz + g(y,z)$ 2

$\Rightarrow \frac{\partial f}{\partial y} = xz + \frac{\partial g}{\partial y}$ 1 $\quad$ 1 $\Rightarrow \frac{\partial g}{\partial y} = -2y\ln(z)$ 1

$= xz - 2y\ln(z)$

$\Rightarrow g = -y^2\ln(z) + h(z)$ 2

$\Rightarrow f = xyz - y^2\ln z + h(z)$ 1

$\Rightarrow \frac{\partial f}{\partial z} = xy - \frac{y^2}{z} + h'(z)$ $\quad \Rightarrow h'(z) = 0 \Rightarrow h(z) = C$ 1

$= xy - \frac{y^2}{z}$ 1

$\boxed{f(x,y,z) = xyz - y^2\ln z + C}$ 1

b). Start point on C: $\vec{r}(0) = \langle 0, 0, 1\rangle$ 2

End point on C: $\vec{r}(1) = \langle 1, 1, e\rangle$ 2

$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = f(1,1,e) - f(0,0,1)$ 1

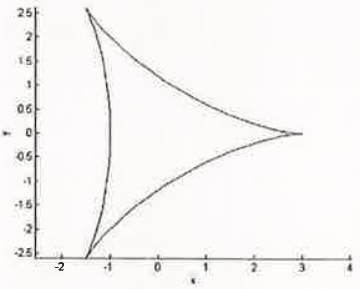
$= (e-1) - (0-0)$ 1

$= \boxed{e-1}$

5. [18 points] Compute the area enclosed by the deltoid curve, pictured below, and parametrized by:

$$\mathbf{r}(\theta) = \langle 2\cos\theta + \cos(2\theta), 2\sin\theta - \sin(2\theta)\rangle, \quad 0 \le \theta \le 2\pi.$$

Reminders: $\sin(2\theta) = 2\sin\theta\cos\theta$ and $\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$.



Green: $A = \frac{1}{2}\oint_C x\,dy - y\,dx$ 1

$x = 2\cos\theta + \cos(2\theta)$ 2

$y = 2\sin\theta - \sin(2\theta)$ 2

$dx = \big(-2\sin\theta - 2\sin(2\theta)\big)\,d\theta$ 2

$dy = \big(2\cos\theta - 2\cos(2\theta)\big)\,d\theta$ 2

4

$$\text{Area} = \frac{1}{2}\int_0^{2\pi} \big(2\cos\theta + \cos(2\theta)\big)\big(2\cos\theta - 2\cos(2\theta)\big) + \big(2\sin\theta - \sin(2\theta)\big)\big(+2\sin\theta + 2\sin(2\theta)\big)\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\Big(\underbrace{4\cos^2\theta} - 4\cos\theta\cos(2\theta) + 2\cos\theta\cos(2\theta) - \underbrace{2\cos^2(2\theta)}$$

$$+ \underbrace{4\sin^2\theta}_{4} + 4\sin\theta\sin(2\theta) - 2\sin\theta\sin(2\theta) - \underbrace{2\sin^2(2\theta)}_{-2}\Big)$$

4

$$= \frac{1}{2}\int_0^{2\pi}\big(2 - 2\cos\theta\cos(2\theta) + 2\sin\theta\sin(2\theta)\big)\,d\theta$$

$$= \frac{1}{2}\left(4\pi - 2\int_0^{2\pi}\Big(\cos\theta\big(1 - 2\sin^2\theta\big) - 2\sin^2\theta\cos\theta\Big)\,d\theta\right)$$

$$= 2\pi - \int_0^{2\pi}\big(\cos\theta - 4\cos\theta\sin^2\theta\big)\,d\theta$$

$$= 2\pi - \left(\sin\theta - 4\frac{\sin^3\theta}{3}\right)\Big|_0^{2\pi}$$

1 $= \boxed{2\pi}$