Name: Solutions

April 13th, 2016.
Math 2551; Sections L1, L2, L3.
Georgia Institute of Technology
Exam 3

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community. By signing my name below I pledge that I have neither given nor received help on this exam.

Pledged: ______________________

Problem	Possible Score	Earned Score
1	16	
2	16	
3	18	
4	18	
5	14	
6	18	
Total	100	

Remember that you must SHOW YOUR WORK to receive credit!

Good luck!

1. The following field:

$$\vec{F}(x,y,z) = \left\langle y^3\cos(xy^3),\ \ 3xy^2\cos(xy^3) + e^{z^2},\ \ 2yze^{z^2} \right\rangle$$

is conservative.

a). Find a potential function f for this field.

$f_x = y^3\cos(xy^3) \Rightarrow f = \sin(xy^3) + g(y,z)$

$\Rightarrow f_y = 3xy^2\cos(xy^3) + g_y \Rightarrow g_y = e^{z^2}$

$\Rightarrow g(y,z) = ye^{z^2} + h(z)$

$\Rightarrow f = \sin(xy^3) + ye^{z^2} + h(z)$

$\Rightarrow f_z = 2yze^{z^2} + h'(z) \Rightarrow h'(z) = 0$

$\Rightarrow \boxed{f = \sin(xy^3) + ye^{z^2} + C}$

(10 pts.)

b). Compute $\int_C \vec{F}\cdot d\vec{r}$, where C is a smooth path from $(\pi, 1, 0)$ to $(0, 2, 1)$.

$$\int_{(\pi,1,0)}^{(0,2,1)} \vec{F}\cdot d\vec{r} = f(0,2,1) - f(\pi,1,0)$$

$$= (2e) - (\sin(\pi) + e^0)$$

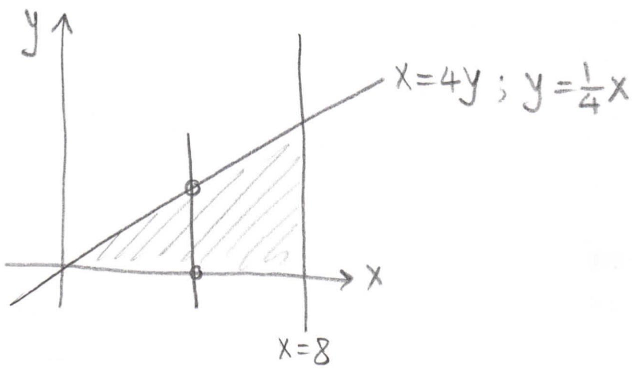
$$= \boxed{2e - 1}$$

(6 pts.)

2. Compute the integral:

$$\int_0^1 \int_0^2 \int_{4y}^8 \frac{\cos(x^2)}{\sqrt{z}}\, dx\, dy\, dz.$$

Switch order of integration between x & y → ③ pts.



$$\int_0^1 \int_0^8 \int_0^{\frac{1}{4}x} \frac{\cos(x^2)}{\sqrt{z}}\, dy\, dx\, dz \quad \rightarrow ⑥ \text{ pts.}$$

$$= \int_0^1 \int_0^8 \frac{\cos(x^2)}{\sqrt{z}} \cdot \frac{1}{4}x \, dx dz \quad \rightarrow ② \text{ pts.}$$

$$= \int_0^1 \frac{1}{\sqrt{z}} \left. \frac{1}{8}\sin(x^2) \right|_{x=0}^{x=8} dz \quad \rightarrow ② \text{ pts.}$$

$$= \int_0^1 \frac{1}{\sqrt{z}} \frac{1}{8}\sin(64)\, dz$$

$$= \left. \frac{1}{4}\sin(64)\sqrt{z} \right|_0^1 \quad \rightarrow ② \text{ pts.}$$

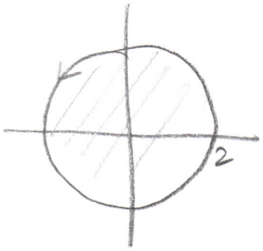
$$= \boxed{\frac{1}{4}\sin(64)} \quad \rightarrow ① \text{ pt.}$$

3. (a). Compute

$$\oint_C \underbrace{(2y+\sqrt{1+x^5})}_{M}\,dx + \underbrace{(5x - e^{y^2})}_{N}\,dy$$

(8 pts.)

where C is the positively oriented circle $x^2+y^2=4$.



Green's Theorem : $\displaystyle\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$

$$= \iint_R (5-2)\,dA$$

$$= 3 \cdot \text{Area}(R)$$

$$= 3 \cdot \pi \cdot 4$$

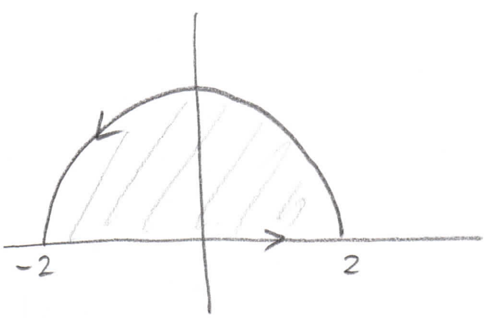
$$= 12\pi$$

(b). Compute the outward flux of the field

$$\mathbf{F}(x,y) = \langle x^2, y^2 \rangle$$

(10 pts.)

across the positively oriented closed curve C, consisting of the upper half of the circle $x^2+y^2=4$, followed by the line segment joining $(-2,0)$ and $(2,0)$ along the x-axis.



$$\oint_C \vec{F}\cdot\vec{n}\,ds = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA$$

$$= \iint_R (2x+2y)\,dA$$

$$= 2\int_0^{\pi}\int_0^{2} (r\cos\theta + r\sin\theta)\, r\,dr\,d\theta$$

$$= 2\int_0^{\pi}\int_0^{2} r^2(\cos\theta+\sin\theta)\,dr\,d\theta$$

$$= 2\int_0^{\pi} \frac{r^3}{3}(\cos\theta+\sin\theta)\Big|_{r=0}^{2}\,d\theta$$

$$= \frac{16}{3}\int_0^{\pi}(\cos\theta+\sin\theta)\,d\theta$$

$$= \frac{16}{3}(\sin\theta-\cos\theta)\Big|_0^{\pi}$$

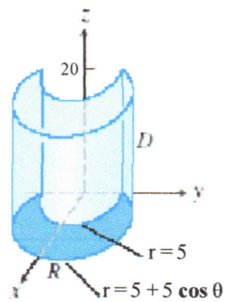
$$= \frac{16}{3}\left((0-(-1))-(0-1)\right)$$

$$= \boxed{\frac{32}{3}}$$

4. (a). Use cylindrical coordinates to set up the triple integral which gives the volume of the solid vertical cylinder whose:

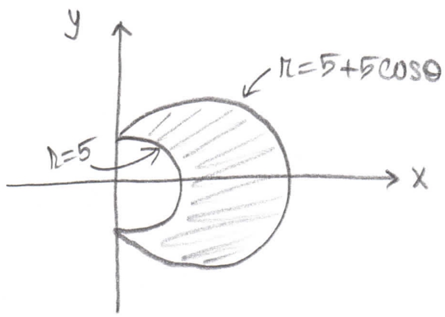
- base is the region in the xy-plane that lies inside the cardioid $r = 5 + 5\cos\theta$ and outside the circle $r = 5$,
- top lies in the plane $z = 20$.

You do not need to compute the volume.

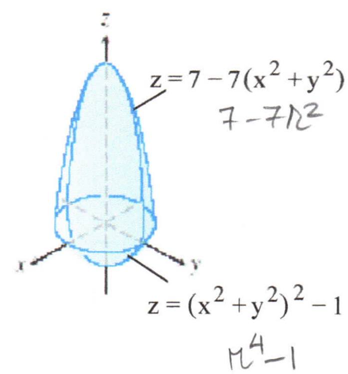


$$V = \int_{-\pi/2}^{\pi/2} \int_{5}^{5+\cos\theta} \int_{0}^{20} r\,dz\,dr\,d\theta$$

(9pts.)



(b). Use cylindrical coordinates to set up the triple integral which gives the volume of the solid bounded by the surfaces $z = 7 - 7(x^2 + y^2)$ and $z = (x^2 + y^2)^2 - 1$. You do not need to compute the volume.



$$\int_{0}^{2\pi} \int_{0}^{1} \int_{r^4-1}^{7-7r^2} r\,dz\,dr\,d\theta$$

(9pts.)

Intersection: $r^4 - 1 = 7 - 7r^2$

$r^4 + 7r^2 - 8 = 0$

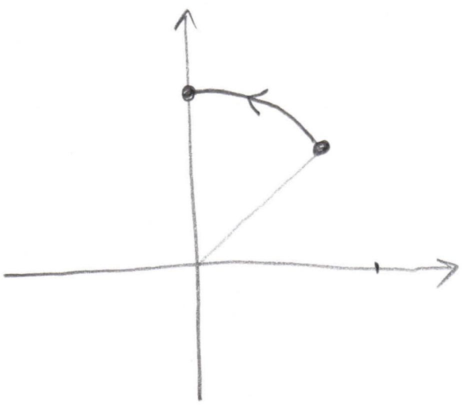
$(r^2+8)(r^2-1) = 0$

$r = 1$

5. Compute the line integral

$$\int_C (x^2 - y)\, ds,$$

where C is the portion of the circle $x^2 + y^2 = 4$ from $(\sqrt{2}, \sqrt{2})$ to $(0, 2)$.



$$C: \begin{cases} x = 2\cos\theta \\ y = 2\sin\theta \end{cases} \qquad \pi/4 \leq \theta \leq \pi/2$$

(3 pts.) (3 pts.)

$$\int_C (x^2 - y)\,ds = \int_{\pi/4}^{\pi/2} \left(4\cos^2\theta - 2\sin\theta\right)\cdot 2\,d\theta$$ (2pts.)

$$= 2\int_{\pi/4}^{\pi/2} 2\left(\cos(2\theta)+1\right) - 2\sin\theta\, d\theta$$ (2pts.)

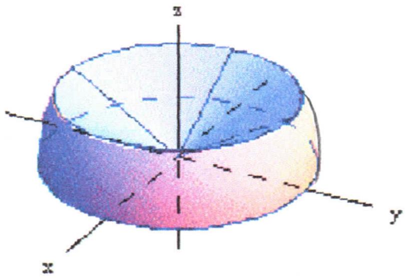
$$= 2\left(\sin(2\theta) + 2\theta + 2\cos\theta\right)\Big|_{\pi/4}^{\pi/2}$$ (2pts.)

$$= 2\left((0+\pi+0) - (1+\pi/2+\sqrt{2})\right)$$

$$= 2\left(\pi/2 - 1 - \sqrt{2}\right)$$ (2pts.)

$$= \boxed{\pi - 2 - 2\sqrt{2}}$$

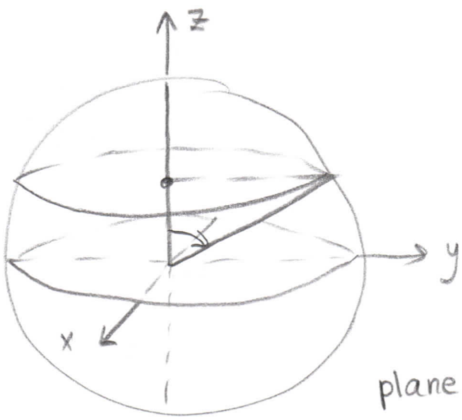
6. (a). Use spherical coordinates to set up the integral which gives the volume of the solid bounded below by the xy-plane, bounded laterally by the sphere $\rho = 8$, and bounded above by the cone $\phi = \frac{\pi}{6}$. You do not need to compute the volume.

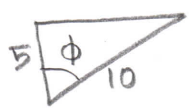


$$\int_0^{2\pi}\int_{\pi/6}^{\pi/2}\int_0^{8} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

(9 pts.)

(b). Use spherical coordinates to set up the integral which gives the volume of the smaller cap cut by $z = 5$ from the sphere $x^2 + y^2 + z^2 = 100$. You do not need to compute the volume.





$\cos\phi = \frac{1}{2}$

$\Rightarrow \phi = \pi/3$

plane $z = 5$:

$\rho\cos\phi = 5$

$\rho = 5/\cos\phi$

$$\int_0^{2\pi}\int_0^{\pi/3}\int_{5/\cos\phi}^{10} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

(9 pts.)

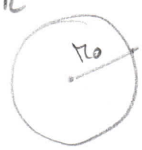
Bonus - 5 points: Set up the volume integral in part (b) above in *cylindrical coordinates.*

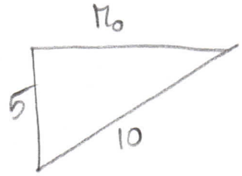
Sphere: $x^2+y^2+z^2=100$

$r^2+z^2=100$

$z=\sqrt{100-r^2}$

Shadow region:





$r_0 = 5\sqrt{3}$

$$\int_0^{2\pi}\int_0^{5\sqrt{3}}\int_5^{\sqrt{100-r^2}} r \, dz \, dr \, d\theta$$