14.1 Functions of Several Variables

- Domain & Range
- Interior Point, Boundary Point
- Open Set, Closed Set, Bounded Set
- Level Curves : $\boxed{f(x,y)=c}$
- Level Surfaces: $\boxed{f(x,y,z)=c}$

14.2 Limits & Continuity in Higher Dimensions

- Major difference from Calculus I limits (one variable): a point (x,y) can approach a point (x_0,y_0) in the plane from infinitely many directions, along infinitely many paths. They must all agree in order for the limit to exist!
- Two-Path Test: If $f(x,y)$ has 2 different limits along 2 different paths in the domain as (x,y) approaches (x_0,y_0), then $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist.

- Typical Examples:

1. Continuous functions (aka "plug it in, nothing bad happens"):

$$\lim_{(x,y)\to(0,1)} \frac{x-xy+3}{x^2y+5xy-y^3} = \boxed{-3}$$

2. 0/0 limits where we factor & cancel terms

$$\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}} = \lim_{(x,y)\to(0,0)} \frac{x(x-y)}{\sqrt{x}-\sqrt{y}} = \lim_{(x,y)\to(0,0)} \frac{x\cancel{(\sqrt{x}-\sqrt{y})}(\sqrt{x}+\sqrt{y})}{\cancel{\sqrt{x}-\sqrt{y}}}$$
$$= \boxed{0}.$$

3. Two-Path Test (Limit DNE):

$f(x,y) = \dfrac{2x^2y}{x^4+y^2}$ has no limit as $(x,y)\to(0,0)$

(approach along lines) $$f(x,y)\Big|_{y=kx} = \frac{2x^2(kx)}{x^4+(kx)^2} = \frac{2kx^3}{x^4+k^2x^2} = \frac{2kx}{x^2+k^2} \xrightarrow[x\to 0]{} \boxed{0}$$

inconclusive

(approach along parabolas) $$f(x,y)\Big|_{y=kx^2} = \frac{2x^2(kx^2)}{x^4+(kx^2)^2} = \frac{2kx^4}{x^4+k^2x^4} = \frac{2k}{1+k^2} \xrightarrow[x\to 0]{} \boxed{\frac{2k}{1+k^2}}$$

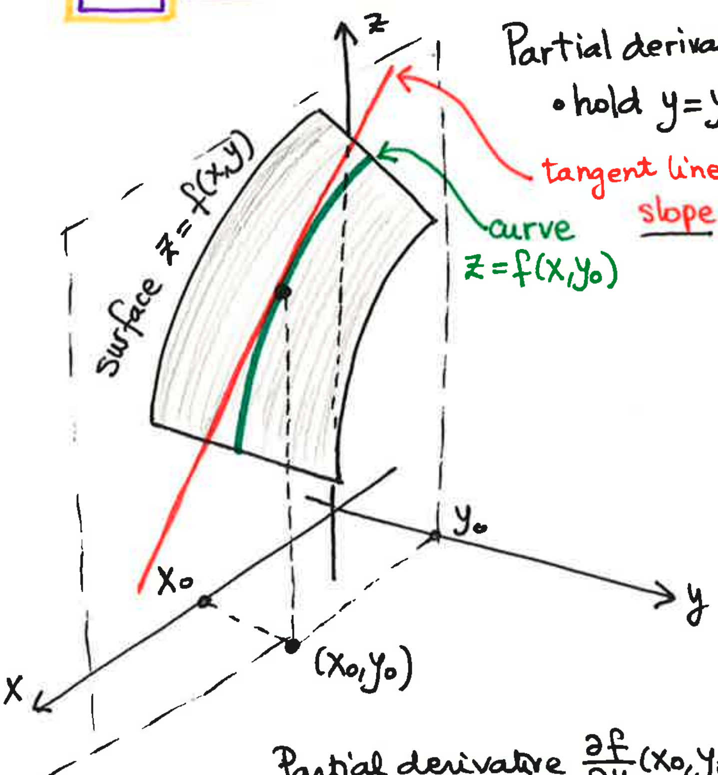
Taking $k=0$ and $k=2$, for example, $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE by the 2-Path Test.

④ Switching to Polar Coord. (to show that a limit does exist).

- $$\lim_{(x,y)\to(0,0)}\left(xy\,\frac{x^2-y^2}{x^2+y^2}\right)=\lim_{r\to 0}\left((r\cos\theta)(r\sin\theta)\,\frac{(r\cos\theta)^2-(r\sin\theta)^2}{(r\cos\theta)^2+(r\sin\theta)^2}\right)$$
$$=\lim_{r\to 0} r^2\underbrace{\cos\theta\sin\theta\,(\cos^2\theta-\sin^2\theta)}_{\text{bounded}}$$
$$=\boxed{0}.$$

- $$\lim_{(x,y)\to(0,0)}\left[\ln\left(\frac{7x^2-x^2y^2+7y^2}{x^2+y^2}\right)\right]=\lim_{r\to 0}\left[\ln\frac{7r^2\cos^2\theta-r^4\cos^2\theta\sin^2\theta+7\sin^2\theta\, r^2}{r^2\cos^2\theta+r^2\sin^2\theta}\right]$$
$$=\lim_{r\to 0}\left[\ln\frac{7r^2(\cos^2\theta+\sin^2\theta)-r^4\cos^2\theta\sin^2\theta}{r^2}\right]$$
$$=\lim_{r\to 0}\left[\ln\left(7-\underbrace{r^2\underbrace{\cos^2\theta\sin^2\theta}_{\text{bounded}}}_{\xrightarrow{r\to 0}\ 0}\right)\right]=\boxed{\ln(7)}$$

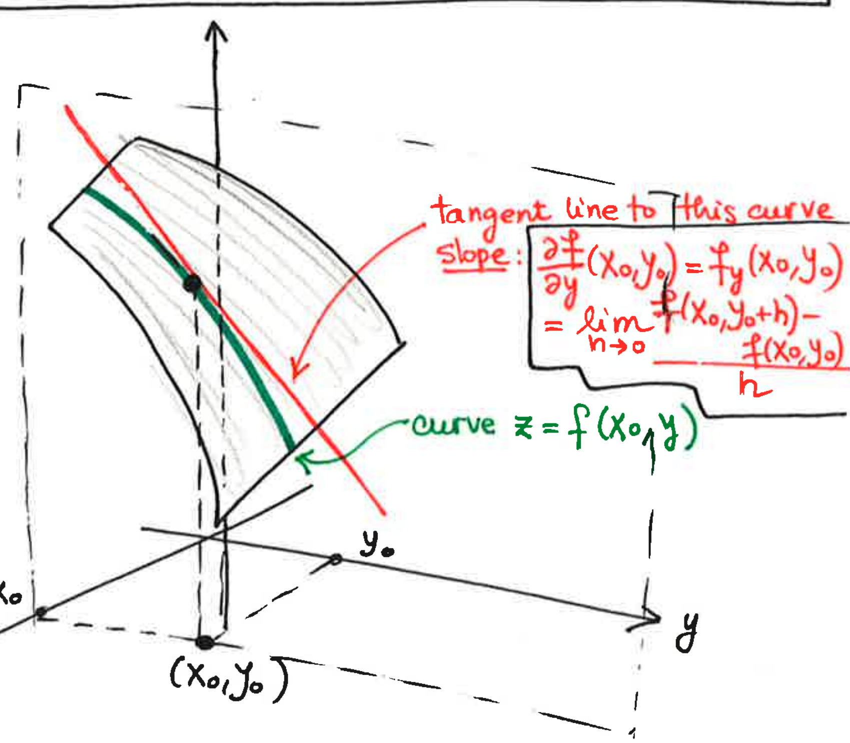
14.3 Partial Derivatives



Partial derivative $\frac{\partial f}{\partial x}(x_0,y_0)$:
- hold $y=y_0$ constant

slope: $$\boxed{\frac{\partial f}{\partial x}(x_0,y_0)=f_x(x_0,y_0)=\lim_{h\to 0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}}$$

Partial derivative $\frac{\partial f}{\partial y}(x_0,y_0)$:
- hold $x=x_0$ constant



- To compute $\frac{\partial f}{\partial x}$: hold the other variables (y, z etc.) constant & differentiate by x.

Ex: $f(x,y,z) = x^2y + x\sin(yz) + \cos(xyz)$

To find $\frac{\partial f}{\partial x}$, think of f as: $f(x,y,z) = x^2y + x\sin(yz) + \cos(xyz)$
(green = variable)

$$\frac{\partial f}{\partial x} = 2xy + \sin(yz) + \left(-\sin(xyz)\cdot yz\right)$$

To find $\frac{\partial f}{\partial y}$, think of f as: $f(x,y,z) = x^2y + x\sin(yz) + \cos(xyz)$

$$\frac{\partial f}{\partial y} = x^2 + x\cos(yz)\cdot z + \left(-\sin(xyz)\cdot xz\right)$$

To find $\frac{\partial f}{\partial z}$, think of f as: $f(x,y,z) = x^2y + x\sin(yz) + \cos(xyz)$

$$\frac{\partial f}{\partial z} = x\cos(yz)\cdot y + \left(-\sin(xyz)\cdot xy\right)$$

- Higher Order Partial Derivatives :

$\frac{\partial^2 f}{\partial x^2} = f_{xx}$; $\frac{\partial^2 f}{\partial y^2} = f_{yy}$; $\frac{\partial^2 f}{\partial x \partial y} = f_{yx}$; $\frac{\partial^2 f}{\partial y \partial x} = f_{xy}$;

$\frac{\partial^3 f}{\partial x \partial y \partial z} = f_{zyx}$; $\frac{\partial^4 f}{\partial x \partial y^3} = f_{yyyx}$; etc. etc.

- Mixed Derivative Theorem :

If $f(x,y)$ and its partial derivatives f_x, f_y, f_{xy}, f_{yx} are defined throughout an open region containing a point (x_0,y_0) and are all continuous at (x_0,y_0), then $f_{xy}(x_0,y_0) = f_{yx}(x_0,y_0)$.

14.4 The Chain Rule :

$w = w(x,y,z)$
$x = x(t)$
$y = y(t)$
$z = z(t)$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = w_x \cdot x'(t) + w_y \cdot y'(t) + w_z \cdot z'(t)$$

$w = w(x,y,z)$
$x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} = w_x \cdot x_u + w_y \cdot y_u + w_z \cdot z_u$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} = w_x \cdot x_v + w_y \cdot y_v + w_z \cdot z_v$$

• **Implicit Differentiation Simplified**: If $F(x,y)=0$ defines y implicitly as a differentiable function of x, then $\boxed{\dfrac{dy}{dx} = -\dfrac{F_x}{F_y}}$

Ex1.: $6x^3-7y^2-xy=0$; find $\frac{dy}{dx}$

Old way: $\frac{d}{dx}(6x^3-7y^2-xy) = \frac{d}{dx}(0)$

$$18x^2 - 14y\left(\frac{dy}{dx}\right) - y - x\left(\frac{dy}{dx}\right) = 0$$

$$18x^2 - y = (x+14y)\frac{dy}{dx}$$

$$\frac{18x^2-y}{x+14y} = \frac{dy}{dx}$$

New way: $F(x,y) = 6x^3-7y^2-xy$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{18x^2-y}{-14y-x} = \frac{18x^2-y}{14y+x}.$$

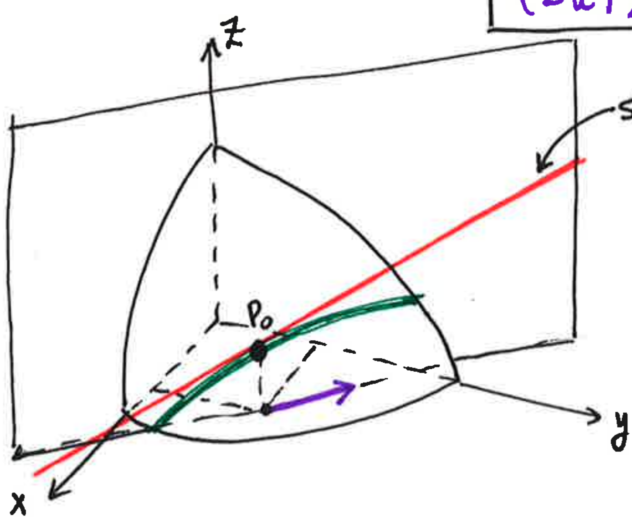
14.5 Directional Derivatives & Gradient Vectors

$f(x,y,z)$; Gradient Vector: $\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$

$f(x,y)$; Gradient Vector: $\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$

Directional Derivative of f at P_0 in the direction of the unit vector $\vec{u}$:

$$\boxed{(D_{\vec{u}}f)_{P_0} = (\nabla f)_{P_0} \cdot \vec{u}}$$



slope $= (D_{\vec{u}}f)_{P_0}$ = rate of change in the direction of $\vec{u}$.

$$D_{\vec{u}}f = |\nabla f||\vec{u}|\cos\theta = |\nabla f|\cos\theta$$

• $f \uparrow$ most rapidly in the direction of ∇f $(\theta=0)$; directional deriv. there is $|\nabla f|$.

• $f \searrow$ most rapidly in the direction of $-\nabla f$ $(\theta=\pi)$; directional deriv. there is $-|\nabla f|$.

14.6 Tangent Planes & Normal Lines

Tangent plane to $f(x,y,z)=c$ at $P_0(x_0,y_0,z_0)$: $\boxed{f_x(P_0)(x-x_0)+f_y(P_0)(y-y_0)+f_z(P_0)(z-z_0)=0}$

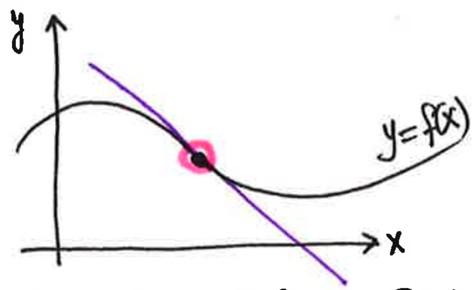
Normal line to $f(x,y,z)=c$ at $P_0(x_0,y_0,z_0)$:

$$\boxed{\begin{aligned} x &= x_0+f_x(P_0)t \\ y &= y_0+f_y(P_0)t \\ z &= z_0+f_z(P_0)t \end{aligned}}$$

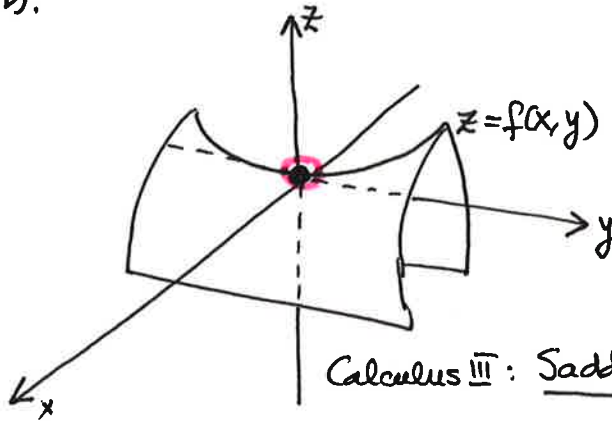
14.7 Extreme Values & Saddle Points

Def.: A point (a,b) is called a critical point of $f(x,y)$ if (a,b) is an interior point to the domain of f and either $f_x(a,b) = f_y(a,b) = 0$ or one or both f_x, f_y do not exist at (a,b).

Def.: A saddle point for $f(x,y)$ is a critical point (a,b) such that in every open disk centered at (a,b) there are domain points (x,y) such that $f(x,y) > f(a,b)$ and points where $f(x,y) < f(a,b)$.



Calculus I & II: Inflection Point



Calculus III: Saddle Point

First Derivative Test:

If $f(x,y)$ has a local min. or max at an interior point (a,b) and $f_x(a,b)$, $f_y(a,b)$ exist, then $f_x(a,b) = f_y(a,b) = 0$.

In other words: local extrema occur at critical points or at boundary points.

Second Derivative Test:

Suppose $f_x(a,b) = f_y(a,b) = 0$ at an interior point (a,b). Then (a,b) is a:

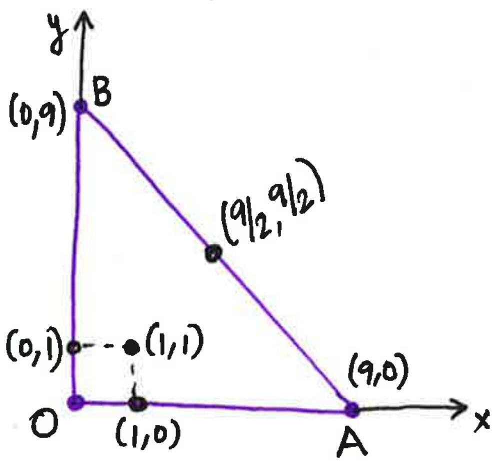
(a,b):

- local extremum if $\boxed{f_{xx} f_{yy} - f_{xy}^2 > 0}$ at (a,b):
  - local max. if $f_{xx} < 0$ at (a,b)
  - local min. if $f_{xx} > 0$ at (a,b)
- saddle point if $\boxed{f_{xx} f_{yy} - f_{xy}^2 < 0}$ at (a,b).
- ? if $\boxed{f_{xx} f_{yy} - f_{xy}^2 = 0}$

Min & Max on Closed Bounded Regions:

1. Find the critical points interior to the region & evaluate f there.
2. Find the boundary points where f has local extrema & evaluate f there.
3. Look through the lists & find the absolute min & max.

Ex: $f(x,y) = 2+2x+2y-x^2-y^2$; Triangular region in Quad I bounded by $x=0$, $y=0$, $y=9-x$.



List:

$\boxed{f(1,1)=4}$ → Max

$f(1,0)=3$

$f(0,0)=2$

$\boxed{f(9,0)=-61}$ → Min

$f(0,1)=3$

$f(0,9)=-61$

$f(9/2,9/2) = -\frac{41}{2}$

1. Interior points: $f_x = 2-2x$; $f_y = 2-2y$

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \quad \begin{cases} 2-2x=0 \\ 2-2y=0 \end{cases} \quad \begin{cases} x=1 \\ y=1 \end{cases}$$

Critical point: $(1,1)$

Is it inside the region? Yes. So evaluate f & list it.

2. Boundary Points:

(i) OA: $y=0$

$f(x,0) = 2+2x-x^2$, $0 \le x \le 9$

Extreme values occur @ boundary pts. or critical points in $(0,9)$

$f'(x,0) = 2-2x$

Critical point: $(1,0)$ – evaluate f here and at the boundary points $(0,0)$, $(9,0)$.

(ii) OB: $x=0$

$f(0,y) = 2+2y-y^2$, $0 \le y \le 9$

$f'(0,y) = 2-2y$

Critical point: $(0,1)$; Boundary points: $(0,0)$, $(0,9)$

↑ already done

(iii) AB: $y=9-x$

Already evaluated endpoints $(0,9)$ & $(9,0)$, so we only need to look for critical points.

$$f(x,9-x) = 2+2x+2(9-x)-x^2-(9-x)^2$$
$$= -61+18x-2x^2$$

$f'(x,9-x) = 18-4x$

$18-4x=0 \Rightarrow x = 9/2 \Rightarrow y = 9-9/2 = 9/2$

Critical point: $(9/2, 9/2)$

$\boxed{14.8}$ Lagrange Multipliers

Suppose that $f(x,y,z)$ and $g(x,y,z)$ are differentiable and $\nabla g \neq \vec{0}$ when $g(x,y,z)=0$.
To find local min & max values of f subject to the constraint $g(x,y,z)=0$

find the values of x,y,z and λ that satisfy:

$$\boxed{\begin{cases} \nabla f = \lambda(\nabla g) \\ g(x,y,z)=0 \end{cases}}$$

Two constraints: To find local extrema of a differentiable $f(x,y,z)$ subject to the constraints $g_1(x,y,z)=0$ and $g_2(x,y,z)=0$ and g_1, g_2 are differentiable (∇g_1 not parallel to ∇g_2), find the values of $x,y,z,\lambda_1,\lambda_2$ that satisfy:

$$\begin{cases} \nabla f = \lambda_1(\nabla g_1) + \lambda_2(\nabla g_2) \\ g_1(x,y,z)=0 \\ g_2(x,y,z)=0 \end{cases}$$

Example: Find the max. value of $f(x,y) = 58 - x^2 - y^2$ on the line $x+7y=50$.

$g(x,y) = x+7y-50$

$\nabla f = \langle -2x, -2y \rangle$

$\nabla g = \langle 1, 7 \rangle$

Solve: $\begin{cases} -2x = \lambda \\ -2y = 7\lambda \\ x+7y-50=0 \end{cases}$ $\quad \begin{cases} x = -\lambda/2 \\ y = -7\lambda/2 \\ x+7y-50=0 \end{cases}$

$-\frac{\lambda}{2} + 7\left(-\frac{7\lambda}{2}\right) - 50 = 0$

$\lambda + 49\lambda + 100 = 0$

$50\lambda = -100$

$\boxed{\lambda = -2}$

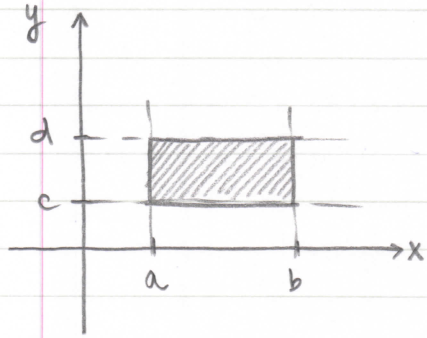
$\Rightarrow \boxed{x = +1, \ y = +7}$

The extreme value occurs at $(1,7)$, where:

$f(1,7) = \boxed{8}$.

Chapter 15 Review

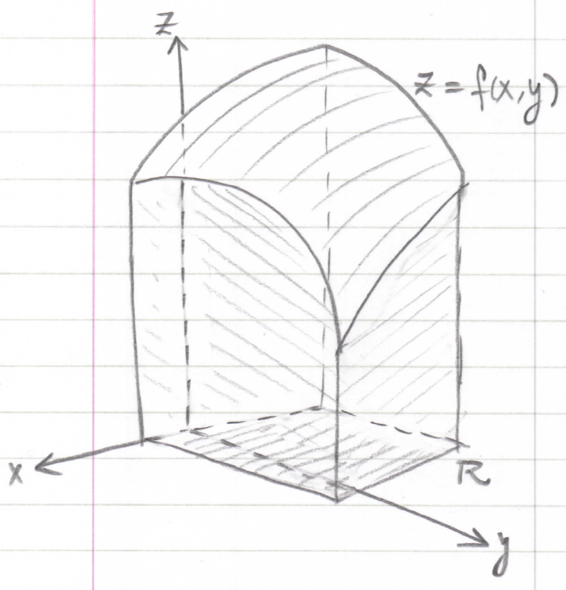
15.1 Double Integrals over Rectangles



- Rectangle R: $a \le x \le b;\ c \le y \le d.$
- Integral of a continuous function $f(x,y)$ over R:

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy$$
$$= \int_a^b \int_c^d f(x,y)\,dy\,dx$$

(Fubini's Theorem).

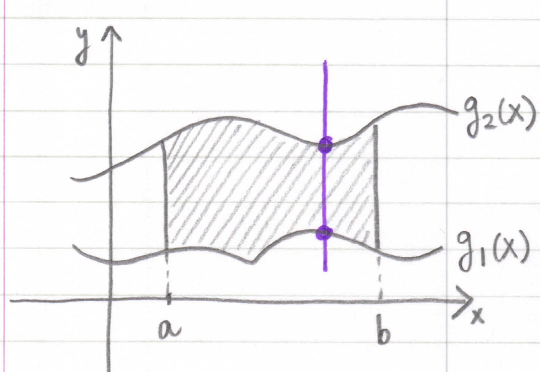


- Volume of solid bounded above by surface $z = f(x,y)$ and below by a rectangle R in the xy-plane:

$$\boxed{V = \iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx}$$

15.2 Double Integrals over "General" Regions:

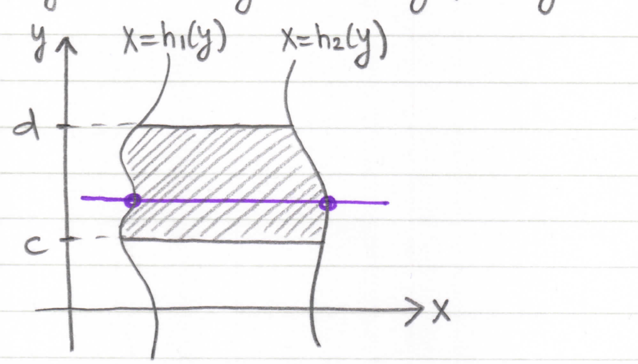
Region R: $a \le x \le b,\ g_1(x) \le y \le g_2(x)$



$$\boxed{\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx}$$

(Vertical Cross-Sections)

Region R: $h_1(y) \le x \le h_2(y),\ c \le y \le d$



$$\boxed{\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy}$$

(Horizontal Cross-Sections).

$\boxed{15.3}$ Area by Double Integration :

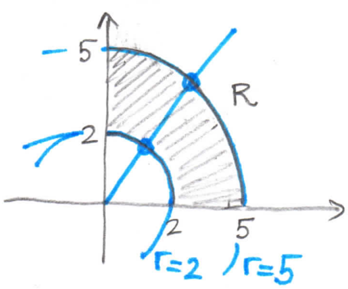
- Area of a closed bounded region R in the plane: $\boxed{\iint_R dA}$
- Average value of $f(x,y)$ over R: $\boxed{\dfrac{1}{\text{area}(R)} \iint_R f(x,y)\,dA}$

$\boxed{15.4}$ Double Integrals in Polar Coordinates:

Area differential $\boxed{dA}$ → $\boxed{dx\,dy}$ or $\boxed{dy\,dx}$ in Cartesian coordinates

→ $\boxed{r\,dr\,d\theta}$ in polar coordinates

Example: $\iint_R xy\,dA = \int_0^{\pi/2} \int_2^5 (r\cos\theta)(r\sin\theta)\, r\,dr\,d\theta$



$$= \int_0^{\pi/2} \int_2^5 r^3 \sin\theta\cos\theta\, dr\,d\theta$$

$$= \int_0^{\pi/2} \frac{r^4}{4} \cdot \frac{1}{2}\sin(2\theta) \Big|_{r=2}^{r=5} d\theta$$

$$= \int_0^{\pi/2} \frac{609}{8} \sin(2\theta)\,d\theta = -\frac{609}{8} \cdot \frac{1}{2}\cos(2\theta)\Big|_0^{\pi/2}$$

$$= -\frac{609}{16}(-1) + \frac{609}{16} = \boxed{\frac{609}{8}}$$