Name: Solutions

February 10$^{\text{th}}$, 2016.
Math 2551; Sections L1, L2, L3.
Georgia Institute of Technology
EXAM 1

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community. By signing my name below I pledge that I have neither given nor received help on this exam.

Pledged: ______________________

Problem	Possible Score	Earned Score
1	20	
2	16	
3	16	
4	18	
5	20	
6	10	
Total	100	

Remember that you must SHOW YOUR WORK to receive credit!

Good luck!

Angle θ $(0 \leq \theta \leq \pi)$ between vectors $\mathbf{u}$ and $\mathbf{v}$:

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}; \quad \sin\theta = \frac{|\mathbf{u}\times\mathbf{v}|}{|\mathbf{u}||\mathbf{v}|}.$$

Vector Projection of $\mathbf{u}$ onto $\mathbf{v} \neq 0$:

$$\mathrm{Proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v} = |\mathbf{u}\cos\theta|\frac{\mathbf{v}}{|\mathbf{v}|}.$$

Distance from a point S to a line L going through P and parallel to $\mathbf{v}$:

$$d = \frac{|\overrightarrow{PS}\times\mathbf{v}|}{|\mathbf{v}|}$$

Length of a smooth curve C: $\mathbf{r}(t)$, traced exactly once as $a \leq t \leq b$:

$$L = \int_a^b |\mathbf{v}(t)|\,dt.$$

Arclength parameter:

$$s(t) = \int_{t_0}^t |\mathbf{v}(\tau)|\,d\tau.$$

TNB Frame:

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}; \quad \mathbf{N} = \frac{d\mathbf{T}/ds}{\kappa} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}; \quad \mathbf{B} = \mathbf{T}\times\mathbf{N}.$$

Curvature:

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}.$$

Tangential and Normal Components of Acceleration:

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N};$$

$$a_T = \frac{d^2s}{dt^2} = \frac{d}{dt}|\mathbf{v}(t)|;$$

$$a_N = \kappa\left(\frac{ds}{dt}\right)^2 = \kappa|\mathbf{v}(t)|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}.$$

Torsion:

$$\tau = -\frac{d\mathbf{B}}{ds}\cdot\mathbf{N} = -\frac{1}{|\mathbf{v}|}\frac{d\mathbf{B}}{dt}\cdot\mathbf{N}.$$

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1. [20 points] Consider the vectors:

$$\mathbf{u} = \langle 1, 0, 2 \rangle,$$
$$\mathbf{v} = \langle -1, 2, 1 \rangle.$$

(4 pts.)

a). Find $\mathbf{u} \cdot \mathbf{v}$.

$$\vec{u} \cdot \vec{v} = -1 + 2 = \boxed{1}$$

(6 pts.)

b). Find the angle θ between the two vectors. Give an exact answer.

$|\vec{u}| = \sqrt{5}$ (2 pts.)

$|\vec{v}| = \sqrt{6}$ (2 pts.)

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \frac{1}{\sqrt{30}} \Rightarrow \boxed{\theta = \cos^{-1}\left(\frac{1}{\sqrt{30}}\right)}$$ (2 pts.)

(6 pts.)

c). Find $\mathbf{u} \times \mathbf{v}$.

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2 \\ -1 & 2 & 1 \end{vmatrix} = \langle -4, -3, 2 \rangle.$$

Correct setup - 3 pts.
Each component - 1 pt.

(4 pts.)

d). Find $\mathbf{v} \times \mathbf{u}$.

$$\vec{v} \times \vec{u} = \langle 4, 3, -2 \rangle.$$

2. [16 points] Find parametric equations for the line tangent to the curve

$$\mathbf{r}(t) = \left\langle e^t,\ te^t,\ te^{t^2} \right\rangle,$$

at the point $(1, 0, 0)$.

$\vec{v}(t) = \left\langle e^t,\ e^t + te^t,\ e^{t^2} + 2t^2 e^{t^2} \right\rangle$ (6pts.)

Value of t that gives $\vec{r}(t) = \langle 1,0,0 \rangle$: $\boxed{t = 0}$ (2pts.)

$\vec{v}(0) = \langle 1, 1, 1 \rangle$ (2pts.)

Parametric Equations:
$$\begin{cases} x = 1 + t \\ y = t \\ z = t \end{cases}$$
(6pts.)

3. [16 points] Find the length of the curve:

$$\mathbf{r}(t) = \langle e^t, e^t \sin t, e^t \cos t \rangle,$$

between the points $(1, 0, 1)$ and $(e^{2\pi}, 0, e^{2\pi})$.

(5 pts.) $\vec{v}(t) = \langle e^t, e^t \sin t + e^t \cos t, e^t \cos t - e^t \sin t \rangle$

$|\vec{v}(t)| = \sqrt{e^{2t}(1 + (\sin t + \cos t)^2 + (\cos t - \sin t)^2)}$

$= e^t \sqrt{1 + \sin^2 t + \cos^2 t + 2\cancel{\sin t \cos t} + \cos^2 t + \sin^2 t - 2\cancel{\sin t \cos t}}$

(5 pts.) $= \underline{\underline{e^t \sqrt{3}}}$

(2 pts.) Value of t that gives $(1,0,1)$: $t=0$

$(e^{2\pi}, 0, e^{2\pi})$: $t = 2\pi$

(4 pts.) $L = \int_0^{2\pi} e^t \sqrt{3}\, dt = \sqrt{3} e^t \Big|_0^{2\pi} = \boxed{\sqrt{3}(e^{2\pi} - 1)}$

4. [18 points] Consider the curve:

$$\mathbf{r}(t) = \langle 3\cos t, 3\sin t, 4t\rangle,$$

and its unit tangent and unit normal vectors:

$$\mathbf{T} = \left\langle -\frac{3}{5}\sin t, \frac{3}{5}\cos t, \frac{4}{5}\right\rangle,$$

$$\mathbf{N} = \langle -\cos t, \sin t, 0\rangle.$$

a). Find the unit binormal vector **B**.

(9pts.) $\vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\frac{3}{5}\sin t & \frac{3}{5}\cos t & \frac{4}{5} \\ -\cos t & -\sin t & 0 \end{vmatrix} = \left\langle +\frac{4}{5}\sin t, -\frac{4}{5}\cos t, +\frac{3}{5}\right\rangle$

(9pts.) b). Find the torsion τ along this curve.

$\frac{d\vec{B}}{dt} = \left\langle \frac{4}{5}\cos t, +\frac{4}{5}\sin t, 0\right\rangle$ (3pts.)

$\vec{v} = \langle -3\sin t, 3\cos t, 4\rangle \Rightarrow |\vec{v}| = 5$ (3pts.)

$\tau = -\frac{1}{|\vec{v}|}\frac{d\vec{B}}{dt}\cdot\vec{N} = -\frac{1}{5}\left(-\frac{4}{5}\cos^2 t - \frac{4}{5}\sin^2 t\right) = \frac{4}{25}$ (3pts.)

5. [20 points] Find the following limits:

(7 pts.) a). $\lim_{(x,y)\to(0,0)} \frac{-2e^{-4y}\sin(3x)}{-x} = \lim_{(x,y)\to(0,0)} \frac{3\cdot 2e^{-4y}\sin(3x)}{3x} = \boxed{6}$

$\xrightarrow{} 1$

(7 pts.) b). $\lim_{\substack{(x,y)\to(0,0)\\ x\neq y}} \frac{x-y+5\sqrt{x}-5\sqrt{y}}{\sqrt{x}-\sqrt{y}} = \lim_{\substack{(x,y)\to(0,0)\\ x\neq y}} \frac{\cancel{(\sqrt{x}-\sqrt{y})}(5+\sqrt{x}+\sqrt{y})}{\cancel{\sqrt{x}-\sqrt{y}}} = 5$

Consider the function

$$f(x,y) = \frac{y^4-2x^2}{y^4+x^2}.$$

(2pts.) c). Find the limit of $f(x,y)$ as (x,y) approaches $(0,0)$ along the x-axis.

$y=0 \Rightarrow \lim_{\substack{(x,y)\to(0,0)\\ y=0}} f(x,y) = -2$

(2pts.) d). Find the limit of $f(x,y)$ as (x,y) approaches $(0,0)$ along the y-axis.

$X=0 \Rightarrow \lim_{\substack{(x,y)\to(0,0)\\ x=0}} f(x,y) = 1$

(2 pts.) e). What conclusion can you draw about

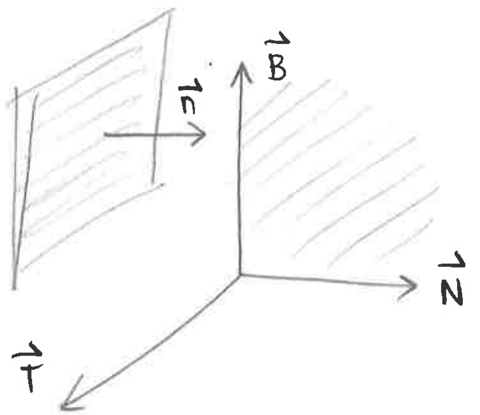
$$\lim_{(x,y)\to(0,0)} \frac{y^4-2x^2}{y^4+x^2}?$$

The limit does not exist by the Two Path Test.

6. [10 points] Find the point on the curve

$$\mathbf{r}(t) = \langle t^3, t^2+1, t-1 \rangle$$

where the normal plane is *orthogonal* to the plane $\frac{1}{3}x - y + z = 4$. Recall that the *normal plane* is determined by **N** and **B**.



Vector normal to $\frac{1}{3}x - y + z = 4$:

$$\boxed{\vec{n} = \langle 1/3, -1, 1 \rangle}$$ (2pts.)

$\vec{n} \perp \vec{T} \Rightarrow \vec{n} \parallel \vec{v}$

$\Rightarrow \vec{n} \cdot \vec{T} = 0 \Rightarrow \vec{n} \cdot \vec{v} = 0$ (3pts.)

$\vec{v}(t) = \langle 3t^2, 2t, 1 \rangle$

$\vec{n} \cdot \vec{v} = t^2 - 2t + 1 \Rightarrow (t-1)^2 = 0 \Rightarrow \boxed{t = 1}$ (4pts.)

$\Rightarrow$ Point: $\boxed{(1, 2, 0)}$ (1pt.)