Math 2401

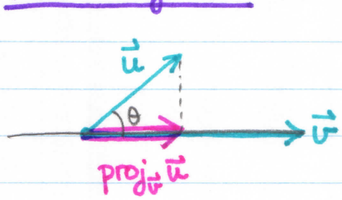
Chapter $\boxed{12}$ Review

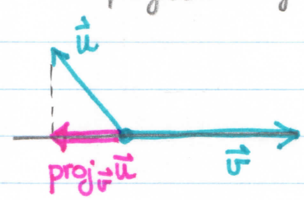
- **Distance formula**: distance between $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$: $\boxed{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}}$
- **Equation of sphere**: with radius R and center (x_0,y_0,z_0): $\boxed{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=R^2}$
- **Magnitude (length) of a vector**: length of $\vec{v}=\langle v_1,v_2,v_3\rangle$: $\boxed{|\vec{v}|=\sqrt{v_1^2+v_2^2+v_3^2}}$
- **Direction of a vector**: direction of $\vec{v}$: $\boxed{\dfrac{\vec{v}}{|\vec{v}|}}$ when $\vec{v}\neq\vec{0}$.
- **Dot Product**: dot product of $\vec{u}=\langle u_1,u_2,u_3\rangle$ and $\vec{v}=\langle v_1,v_2,v_3\rangle$: $\boxed{\vec{u}\cdot\vec{v}=u_1v_1+u_2v_2+u_3v_3}$
- **Angle b/w vectors**: angle θ $(0\leq\theta\leq\pi)$ between $\vec{u}$ and $\vec{v}$: $\boxed{\theta=\cos^{-1}\left(\dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right)}$
- **Dot Product Properties**:
  $\vec{u}\cdot\vec{v}=\vec{v}\cdot\vec{u}$
  $(c\vec{u})\cdot\vec{v}=c(\vec{u}\cdot\vec{v})$
  $\vec{u}\cdot(\vec{v}+\vec{w})=\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$

  $\boxed{\vec{u}\cdot\vec{u}=|\vec{u}|^2}$

  $\boxed{\boxed{\vec{u}\perp\vec{v}} \Leftrightarrow \boxed{\vec{u}\cdot\vec{v}=0}}$

- **Vector Projection**: The vector projection of $\vec{u}$ onto $\vec{v}$ $(\vec{v}\neq 0)$:

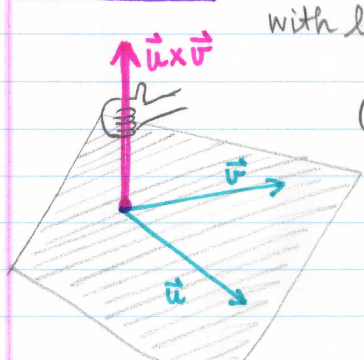




"scalar component of $\vec{u}$ in the direction of $\vec{v}$"

$$\boxed{\begin{aligned}\text{proj}_{\vec{v}}\vec{u}&=\underbrace{\left(|\vec{u}|\cos\theta\right)}\frac{\vec{v}}{|\vec{v}|}\\&=\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v}\end{aligned}}$$

- **Cross Product**: $(\vec{u}\times\vec{v})$ is a vector perpendicular to the plane determined by $\vec{u}$ & $\vec{v}$, with length: $\boxed{|\vec{u}\times\vec{v}|=|\vec{u}||\vec{v}||\sin\theta|}$ → area of parallelogram determined by $\vec{u}$ & $\vec{v}$



(Right Hand Rule)

Determinant Form of Cross Product:

$$\boxed{\vec{u}\times\vec{v}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\u_1&u_2&u_3\\v_1&v_2&v_3\end{vmatrix}}$$

- **Properties of Cross Product**

$$(a\vec{u})\times(b\vec{v}) = (ab)(\vec{u}\times\vec{v})$$

$$\boxed{\vec{v}\times\vec{u} = -(\vec{u}\times\vec{v})}$$

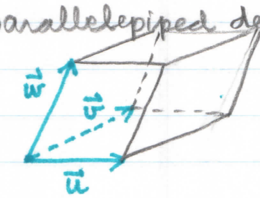
$$\vec{u}\times(\vec{v}+\vec{w}) = \vec{u}\times\vec{v} + \vec{u}\times\vec{w}$$

$$(\vec{u}+\vec{v})\times\vec{w} = \vec{u}\times\vec{w} + \vec{v}\times\vec{w}$$

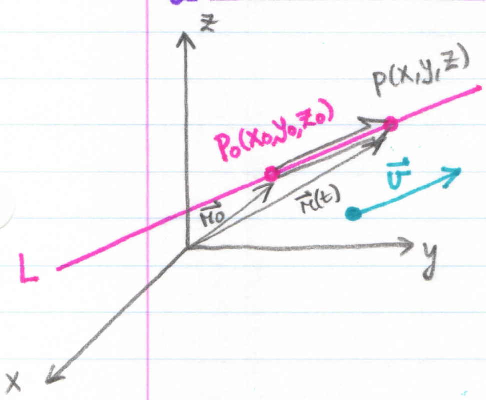
- **Triple Scalar (Box) Product**:

$$\boxed{(\vec{u}\times\vec{v})\cdot\vec{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}}$$

→ Volume of parallelepiped determined by $\vec{u}, \vec{v}, \vec{w}$.



- **Equations for a Line in Space**



L: The line going through the point $P_0(x_0, y_0, z_0)$ and parallel to the vector $\vec{v} = \langle v_1, v_2, v_3\rangle$:

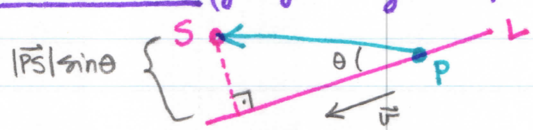
Vector Eqn.: $\boxed{\vec{r}(t) = \vec{r}_0 + t\vec{v}}$

Parametric Eqns.:

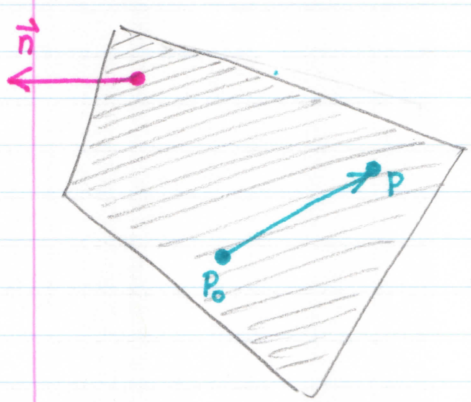
$$\boxed{\begin{aligned} x &= x_0 + tv_1 \\ y &= y_0 + tv_2 \\ z &= z_0 + tv_3 \end{aligned}}$$

- **Distance from a point S to a line (going through P & parallel to $\vec{v}$)**:

$$\boxed{d = \frac{|\vec{PS}\times\vec{v}|}{|\vec{v}|}}$$



- **Equations for a Plane in Space**



The plane going through the point $P_0(x_0, y_0, z_0)$ with a normal vector $\vec{n} = A\vec{i} + B\vec{j} + C\vec{k}$:

Vector Eqn.: $\boxed{\vec{n}\cdot\vec{P_0P} = \vec{0}}$

Component Eqn.: $\boxed{A(x-x_0) + B(y-y_0) + C(z-z_0) = 0}$

Simplified Component Eqn.: $\boxed{Ax + By + Cz = D, \text{ w/ } D = Ax_0 + By_0 + Cz_0}$

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Chapter 13 Review

- **Vector-valued functions / Curves**

$$\boxed{\vec{r}(t) = \langle x(t), y(t), z(t)\rangle = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}}$$ is called

Continuous at t_0 if $\lim_{t\to t_0} \vec{r}(t) = \vec{r}(t_0)$

Differentiable at t_0 if each of the component functions $x(t), y(t), z(t)$ is differentiable at t_0.

Smooth if the derivative $\vec{r}\,'(t)$ is continuous and never $\vec{0}$.

- **Vectors/Quantities associated to $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ (smooth):**

- **Velocity:** $\boxed{\vec{v}(t) = \vec{r}\,'(t) = \frac{d}{dt}\vec{r} = \langle x'(t), y'(t), z'(t)\rangle}$ (derivative of position)

- **Speed:** $\boxed{|\vec{v}(t)|}$

- **Acceleration:** $\boxed{\vec{a}(t) = \frac{d}{dt}\vec{v} = \frac{d^2\vec{r}}{dt^2}}$ (derivative of velocity)

- **Length:** $\boxed{L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt = \int_a^b |\vec{v}(t)|\,dt}$ Length of a smooth curve that is traced exactly once as $a \le t \le b$.

- **Arc Length Parameter:** $\boxed{s(t) = \int_{t_0}^{t} |\vec{v}(\tau)|\,d\tau}$ Length along the curve, measured from a basepoint $P(t_0) = (x(t_0), y(t_0), z(t_0))$.

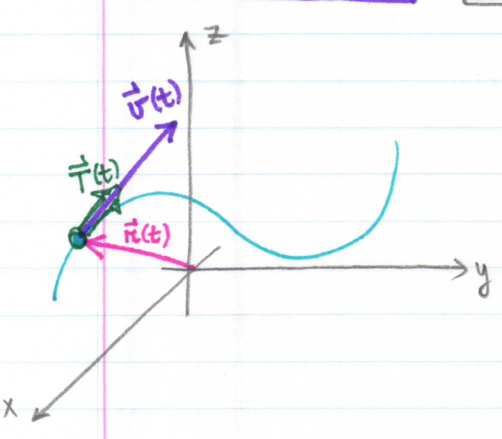
$$\boxed{\frac{ds}{dt} = |\vec{v}(t)|} \qquad \boxed{\frac{dt}{ds} = \frac{1}{|\vec{v}(t)|}}$$

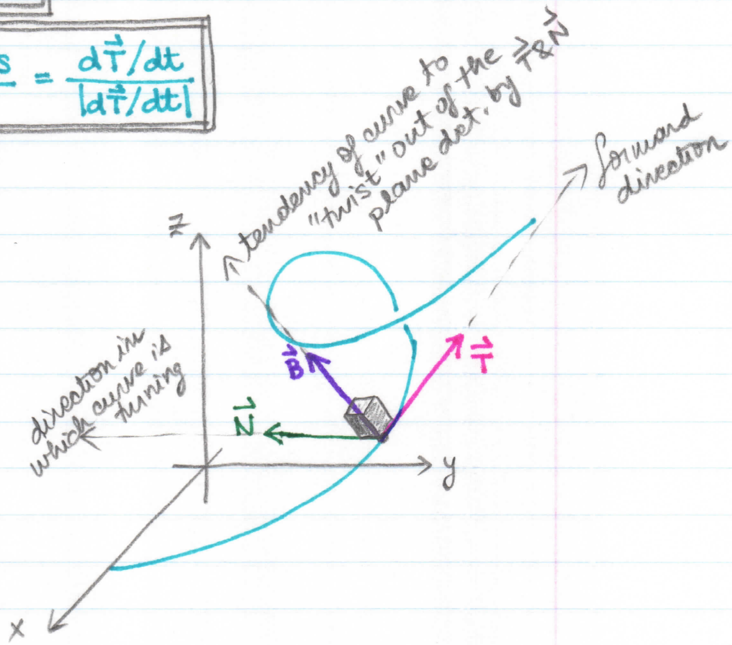
- **Unit Tangent Vector:** $\boxed{\vec{T} = \frac{\vec{v}}{|\vec{v}|}}$; $\boxed{\frac{d\vec{r}}{ds} = \vec{T}}$

- **Curvature:** $\boxed{\kappa = \left|\frac{d\vec{T}}{ds}\right| = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|}$

- **Unit Normal Vector:** $\boxed{\vec{N} = \frac{d\vec{T}/ds}{\kappa} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}}$

- **Binormal Vector:** $\boxed{\vec{B} = \vec{T} \times \vec{N}}$





- **Tangential & Normal Components of Acceleration:**

$$\vec{a} = a_T \vec{T} + a_N \vec{N}$$

$$\boxed{a_T} = \frac{d^2 s}{dt^2} = \boxed{\frac{d}{dt}|\vec{v}(t)|}$$
— Tangential scalar component of acceleration (measures how much of $\vec{a}$ is acting in the direction of motion)

$$\boxed{a_N} = \kappa\left(\frac{ds}{dt}\right)^2 = \kappa|\vec{v}(t)|^2 = \boxed{\sqrt{|\vec{a}|^2 - a_T^2}}$$
→ Normal scalar component of $\vec{a}$ (measures how much of $\vec{a}$ is acting normal to the motion)

- **Torsion:** $\boxed{\tau = -\frac{d\vec{B}}{ds}\cdot\vec{N}}$

- **Differentiation Rules for Vector Functions:**

$$\frac{d}{dt}\left[\vec{u}(t)\cdot\vec{v}(t)\right] = \left(\frac{d}{dt}\vec{u}\right)\cdot\vec{v} + \vec{u}\cdot\left(\frac{d}{dt}\vec{v}\right)$$

$$\frac{d}{dt}\left[\vec{u}(t)\times\vec{v}(t)\right] = \left(\frac{d}{dt}\vec{u}\right)\times\vec{v} + \vec{u}\times\left(\frac{d}{dt}\vec{v}\right)$$

$$\frac{d}{dt}\left[\vec{u}(f(t))\right] = f'(t)\,\vec{u}'(f(t))$$

- **Vector Functions of Constant Length:**

If $\vec{r}(t)$ is a differentiable function of t with constant length, then $\vec{r}(t)$ and $\vec{r}'(t)$ are orthogonal at all t:

$$\vec{r}\cdot\frac{d\vec{r}}{dt} = 0.$$

14.1 Functions of Several Variables

- Domain & Range
- Interior Point, Boundary Point
- Open Set, Closed Set, Bounded Set
- Level Curves : $\boxed{f(x,y)=c}$
- Level Surfaces : $\boxed{f(x,y,z)=c}$

14.2 Limits & Continuity in Higher Dimensions

- Major difference from Calculus I limits (one variable) : a point (x,y) can approach a point (x_0,y_0) in the plane from infinitely many directions, along infinitely many paths. They must all agree in order for the limit to exist!
- Two-Path Test : If $f(x,y)$ has 2 different limits along 2 different paths in the domain as (x,y) approaches (x_0,y_0), then $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist.

- Typical Examples :

(1) Continuous functions (aka "plug it in, nothing bad happens") :

$$\lim_{(x,y)\to(0,1)} \frac{x-xy+3}{x^2y+5xy-y^3} = \boxed{-3}$$

(2) 0/0 limits where we factor & cancel terms

$$\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}} = \lim_{(x,y)\to(0,0)} \frac{x(x-y)}{\sqrt{x}-\sqrt{y}} = \lim_{(x,y)\to(0,0)} \frac{x\cancel{(\sqrt{x}-\sqrt{y})}(\sqrt{x}+\sqrt{y})}{\cancel{\sqrt{x}-\sqrt{y}}} = \boxed{0}.$$

(3) Two-Path Test (Limit DNE) :

$f(x,y) = \dfrac{2x^2y}{x^4+y^2}$ has no limit as $(x,y)\to(0,0)$

(approach along lines) $$f(x,y)\Big|_{y=kx} = \frac{2x^2(kx)}{x^4+(kx)^2} = \frac{2kx^3}{x^4+k^2x^2} = \frac{2kx}{x^2+k^2} \xrightarrow[x\to 0]{} \boxed{0}$$ inconclusive

(approach along parabolas) $$f(x,y)\Big|_{y=kx^2} = \frac{2x^2(kx^2)}{x^4+(kx^2)^2} = \frac{2kx^4}{x^4+k^2x^4} = \frac{2k}{1+k^2} \xrightarrow[x\to 0]{} \boxed{\frac{2k}{1+k^2}}$$

Taking $k=0$ and $k=2$, for example, $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE by the 2-Path Test.

④ Switching to Polar Coord. (to show that a limit does exist).

- $$\lim_{(x,y)\to(0,0)} \left(xy\,\frac{x^2-y^2}{x^2+y^2} \right) = \lim_{r\to 0} \left((r\cos\theta)(r\sin\theta)\,\frac{(r\cos\theta)^2-(r\sin\theta)^2}{(r\cos\theta)^2+(r\sin\theta)^2} \right)$$
$$= \lim_{r\to 0} r^2 \underbrace{\cos\theta\sin\theta\,(\cos^2\theta-\sin^2\theta)}_{\text{bounded}}$$
$$= \boxed{0}.$$

- $$\lim_{(x,y)\to(0,0)} \left[\ln\left(\frac{7x^2-x^2y^2+7y^2}{x^2+y^2} \right) \right] = \lim_{r\to 0} \left[\ln \frac{7r^2\cos^2\theta - r^4\cos^2\theta\sin^2\theta + 7\sin^2\theta\, r^2}{r^2\cos^2\theta + r^2\sin^2\theta} \right]$$
$$= \lim_{r\to 0} \left[\ln \frac{7r^2(\cos^2\theta+\sin^2\theta) - r^4\cos^2\theta\sin^2\theta}{r^2} \right]$$
$$= \lim_{r\to 0} \left[\ln\left(7 - \underbrace{r^2 \underbrace{\cos^2\theta\sin^2\theta}_{\text{bounded}}}_{\xrightarrow{r\to 0}\ 0} \right) \right] = \boxed{\ln(7)}$$