Quiz 8 - Solutions

Grading: 2 pts. each problem
- (1 pt.) for correct conclusi
- (1 pt.) for clear write-up (what test is used, ho et

① $\sum_{n=1}^{\infty} \frac{1}{n+3^n}$

Method 1: Comparison Test

$$\frac{1}{n+3^n} < \frac{1}{3^n}$$

$\sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ is a convergent geometric series (ratio $r = \frac{1}{3} < 1$)

$\Rightarrow$ By the Comparison Test, the series converges.

Method 2: Limit Comparison Test

$$a_n = \frac{1}{n+3^n} \; ; \; b_n = \frac{1}{3^n}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{3^n}{n+3^n} = \lim_{n\to\infty} \frac{1}{\frac{n}{3^n}+1} = 1$$

$\Rightarrow$ By the Limit Comparison Test, since $\sum_{n=1}^{\infty} b_n$ is convergent (geometric series w/ $r = \frac{1}{3} < 1$), $\sum_{n=1}^{\infty} a_n$ is also convergent.

Method 3: Ratio Test

$$a_n = \frac{1}{n+3^n}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1}{n+1+3^{n+1}}(n+3^n) = \lim_{n\to\infty} \frac{n+3^n}{n+1+3^{n+1}}$$

$$= \lim_{n\to\infty} \frac{\frac{n}{3^n}+1}{\frac{n+1}{3^n}+3} = \frac{1}{3} < 1 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ is (absolutely) convergent.}$$

② $\sum_{n=1}^{\infty} \frac{n^2 2^{n-1}}{(-5)^n}$

Ratio Test: $a_n = \frac{n^2 2^{n-1}}{(-5)^n}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^2 2^n}{5^{n+1}} \cdot \frac{5^n}{n^2 \cdot 2^{n-1}}\right| = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 \cdot \frac{2}{5} = \frac{2}{5} < 1$$

$\Rightarrow$ By the Ratio Test, the series is absolutely convergent.

Remark: Could have used AST, but it's more difficult to show b_n is decreasing.

③ $\sum_{n=1}^{\infty} \frac{\sin(2n)}{1+2^n}$

$$a_n = \frac{\sin(2n)}{1+2^n}$$

$$|a_n| = \frac{|\sin(2n)|}{1+2^n} \leq \frac{1}{1+2^n} < \frac{1}{2^n}$$

$\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ is a <u>convergent</u> geometric series ($r = \frac{1}{2} < 1$)

=> By the <u>Comparison Test</u>, $\sum_{n=1}^{\infty} |a_n|$ is convergent => $\sum_{n=1}^{\infty} a_n$ is <u>absolutely convergent</u>.

④ $\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n-1}{2n+1} = \frac{1}{2} \neq 0$ => The series <u>diverges</u> by the <u>Test for Divergence</u>.

⑤ $\sum_{n=1}^{\infty} n e^{-n^2}$

<u>Integral Test</u>: $\int_1^{\infty} x e^{-x^2} dx = -\frac{1}{2} e^{-x^2} \Big|_1^{\infty} = 0 + \frac{1}{2} e^{-1} < \infty$ convergent

=> The series <u>converges</u> by the Integral Test.

<u>Ratio Test</u>: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{n+1}{e^{(n+1)^2}} \frac{e^{n^2}}{n}$

$$= \lim_{n\to\infty} \frac{n+1}{n} \frac{e^{n^2}}{e^{(n+1)^2}} = \lim_{n\to\infty} \frac{n+1}{n} \frac{e^{n^2}}{e^{n^2+2n+1}}$$

$$= \lim_{n\to\infty} \underbrace{\frac{n+1}{n}}_{\downarrow\, 1} \underbrace{\frac{1}{e^{2n+1}}}_{\downarrow\, 0} = 0 < 1$$

=> The series converges by the <u>Ratio Test</u>.