① a). $\sum_{n=1}^{\infty} 5(-1)^n \cdot n x^n$ $R=1$ $I=(-1,1)$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{5(-1)^{n+1}(n+1)x^{n+1}}{5(-1)^n \cdot n \cdot x^n}\right| = \frac{n+1}{n} \cdot |x| \xrightarrow[n\to\infty]{} |x| < 1$$

$x=1 \Rightarrow \sum_{n=1}^{\infty} 5(-1)^n \cdot n$ diverges by the Test for Divergence

$\lim_{n\to\infty} 5(-1)^n \cdot n$ does not exist

$x=-1 \Rightarrow \sum_{n=1}^{\infty} 5(-1)^n \cdot n(-1)^n = \sum_{n=1}^{\infty} 5n(-1)^{2n} = \sum_{n=1}^{\infty} 5n$ diverges by the Test for Divergence

$\lim_{n\to\infty} 5n = \infty$

b). $\sum_{n=1}^{\infty} \frac{x^n}{3n-1}$ $R=1$ $I=[-1,1)$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{3n+2} \cdot \frac{3n-1}{x^n}\right| = \frac{3n-1}{3n+2} \cdot |x| \xrightarrow[n\to\infty]{} |x| < 1$$

$x=1$: $\sum_{n=1}^{\infty} \frac{1}{3n-1}$ diverges by the Comparison Test:

$\frac{1}{3n-1} \geq \frac{1}{3n}$ and $\sum_{n=1}^{\infty} \frac{1}{3n}$ diverges (multiple of the harmonic series)

$x=-1$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n-1}$ converges by the Alternating Series Test

$b_n = \frac{1}{3n-1} \xrightarrow[n\to\infty]{} 0$

$b_{n+1} = \frac{1}{3n+2} \leq \frac{1}{3n-1} = b_n$

c). $\sum_{n=1}^{\infty} \frac{x^{n+2}}{3n!}$ $R=\infty$ $I=(-\infty,\infty)$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+3}}{3(n+1)!} \cdot \frac{3n!}{x^{n+2}}\right| = \frac{|x|}{n+1} \xrightarrow[n\to\infty]{} 0 < 1 \text{ for all } x$$

d). $\sum_{n=1}^{\infty} \frac{7^n \cdot x^n}{n^3}$ $R=\frac{1}{7}$ $I=[-\frac{1}{7},\frac{1}{7}]$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{7^{n+1}\cdot x^{n+1}}{(n+1)^3} \cdot \frac{n^3}{7^n \cdot x^n}\right| = 7\cdot|x|\cdot\frac{n^3}{(n+1)^3} \xrightarrow[n\to\infty]{} 7|x| < 1$$

$$|x| < \frac{1}{7}$$

$x=\frac{1}{7}$: $\sum_{n=1}^{\infty} \frac{1}{n^3}$ convergent p-series

$x=\frac{-1}{7}$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ convergent (because it is absolutely convergent).

e). $\sum_{n=0}^{\infty} \frac{(x-4)^n}{n^5+1}$ $R=1$ $I=[3,5]$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-4)^{n+1}}{(n+1)^5+1} \cdot \frac{n^5+1}{(x-4)^n}\right| = |x-4|\cdot\frac{n^5+1}{(n+1)^5+1} \xrightarrow[n\to\infty]{} |x-4| < 1$$

$$|x-4|<1 \Rightarrow -1 < x-4 < 1 \Rightarrow 3 < x < 5$$

$x=5$: $\sum_{n=0}^{\infty} \frac{1}{n^5+1}$ convergent by the Comparison Test

$\frac{1}{n^5+1} \le \frac{1}{n^5}$ and $\sum_{n=1}^{\infty} \frac{1}{n^5}$ converges (p-series with $p=5>1$)

$x=3$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^5+1}$ convergent because it is absolutely convergent f). $\sum_{n=1}^{\infty} n!\,(x+2)^n$ $R=0$ $I=\{-2\}$

$\lim_{n\to\infty} n!\,(x+2)^n = \infty$ or does not exist for all $x \neq -2$

g). $\sum_{n=1}^{\infty} \frac{n}{2^n}(x+2)^n$ $R=2$ $I=(-4,0)$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)(x+2)^{n+1}}{2^{n+1}} \cdot \frac{2^n}{n(x+2)^n}\right| = \frac{n+1}{2n}|x+2| \xrightarrow[n\to\infty]{} \frac{|x+2|}{2} < 1$$

$$|x+2|<2 \Rightarrow -2<x+2<2 \Rightarrow -4<x<0$$

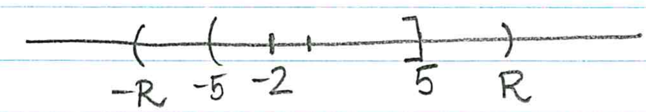
$x=-4$: $\sum_{n=1}^{\infty} \frac{n}{2^n}(-2)^n = \sum_{n=1}^{\infty} (-1)^n \cdot n$ diverges by the Test for Divergence

$\lim_{n\to\infty} (-1)^n \cdot n$ does not exist

$x=0$: $\sum_{n=1}^{\infty} \frac{n}{2^n}\cdot 2^n = \sum_{n=1}^{\infty} n$ diverges by the Test for Divergence

$\lim_{n\to\infty} n = \infty$.

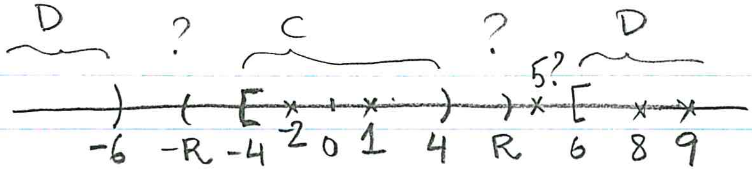
(2) $\sum_{n=1}^{\infty} c_n \cdot 5^n$ converges $\Rightarrow$ $\sum_{n=1}^{\infty} c_n \cdot x^n$ converges at least for $-5<x\leq 5$



a). $\sum_{n=1}^{\infty} c_n(-2)^n$ converges because $-2 \in (-5,5]$

b). $\sum_{n=1}^{\infty} c_n(-5)^n$ – we cannot say

(3)



$\sum_{n=1}^{\infty} C_n(-4)^n$ converges $\Rightarrow$ $\sum_{n=1}^{\infty} C_n X^n$ converges at least on $[-4,4)$

$\sum_{n=1}^{\infty} C_n \cdot 6^n$ diverges $\Rightarrow$ $\sum_{n=1}^{\infty} C_n X^n$ diverges at least on $(-\infty,-6) \cup [6,\infty)$

a). $\sum_{n=1}^{\infty} C_n$ converges $(1 \in [-4,4))$

b). $\sum_{n=1}^{\infty} C_n 8^n$ diverges $(8 \in [6,\infty))$

c). $\sum_{n=1}^{\infty} C_n(-2)^n$ converges $(-2 \in [-4,4))$

d). $\sum_{n=1}^{\infty} C_n \cdot 9^n$ diverges $(9 \in [6,\infty))$

e). $\sum_{n=1}^{\infty} C_n(-6)^n$ cannot say

f). $\sum_{n=1}^{\infty} C_n \cdot 5^n$ cannot say

(4) a). $f(x) = \frac{1}{x+8} = \frac{1}{8} \frac{1}{1-\left(-\frac{x}{8}\right)} = \frac{1}{8} \cdot \sum_{n=0}^{\infty} \left(\frac{-x}{8}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{8^{n+1}} x^n$

for all $\left|\frac{-x}{8}\right| < 1$, so $|x| < 8 \Rightarrow$ $R=8$ $I=(-8,8)$

b). $f(x) = \frac{1}{7-x} = \frac{1}{7} \frac{1}{1-\frac{x}{7}} = \frac{1}{7} \cdot \sum_{n=0}^{\infty} \left(\frac{x}{7}\right)^n = \sum_{n=0}^{\infty} \frac{x^n}{7^{n+1}}$

for all $\left|\frac{x}{7}\right| < 1$, so $|x| < 7 \Rightarrow$ $R=7$ $I=(-7,7)$

c). $f(x) = \frac{x}{4+x^2} = x \cdot \frac{1}{4} \frac{1}{1+\frac{x^2}{4}} = \frac{x}{4} \cdot \frac{1}{1-\left(-\frac{x^2}{4}\right)} = \frac{x}{4} \cdot \sum_{n=0}^{\infty} \left(-\frac{x^2}{4}\right)^n$

$= \frac{x}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n} \cdot x^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^{n+1}} \cdot x^{2n+1}$

for all $\left|-\frac{x^2}{4}\right| < 1$, so $|x^2| < 4$, so $x^2 < 4$, so $|x| < 2$

$R=2$ $I=(-2,2)$

d). $f(x) = \frac{x^2}{1-x^3} = x^2 \cdot \frac{1}{1-x^3} = x^2 \cdot \sum_{n=0}^{\infty} (x^3)^n = \sum_{n=0}^{\infty} x^{3n+2}$

for all $|x^3| < 1$, so $|x| < 1 \Rightarrow$ $R=1$ $I=(-1,1)$

(5) $\ln(3-x) = -\int \frac{1}{3-x}\,dx = -\frac{1}{3}\int \frac{1}{1-\frac{x}{3}}\,dx = -\frac{1}{3}\int \sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n dx$

$= -\frac{1}{3}\sum_{n=0}^{\infty}\int \frac{x^n}{3^n}\,dx = \left(-\frac{1}{3}\sum_{n=0}^{\infty}\frac{1}{3^n}\cdot\frac{x^{n+1}}{n+1}\right) + C$

$= C - \sum_{n=0}^{\infty}\frac{x^{n+1}}{3^{n+1}(n+1)}$ for all $\left|\frac{x}{3}\right| < 1 \Rightarrow |x| < 3$ $R=3$

$x = 0 \Rightarrow \ln(3) = C$

$$\ln(3-x) = \ln(3) - \sum_{n=0}^{\infty}\frac{x^{n+1}}{3^{n+1}(n+1)}$$

(6) $\int \frac{x}{1-x^{11}}\,dx = \int x\cdot\frac{1}{1-x^{11}}\,dx = \int x\cdot\sum_{n=0}^{\infty}(x^{11})^n\,dx$

$= \int \sum_{n=0}^{\infty} x^{11n+1}\,dx = \sum_{n=0}^{\infty}\int x^{11n+1}\,dx$

$= \left(\sum_{n=0}^{\infty}\frac{x^{11n+2}}{11n+2}\right) + C$ for all $|x^{11}| < 1$, or $|x| < 1$ $R=1$

(7) $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for all $|x|<1$

a). $\Rightarrow \sum_{n=1}^{\infty} n\cdot x^{n-1} = \left(\frac{1}{1-x}\right)' \Rightarrow \sum_{n=1}^{\infty} n\cdot x^{n-1} = \boxed{\frac{1}{(1-x)^2}}$

b). $\sum_{n=1}^{\infty} n\cdot x^{n-1} = \frac{1}{(1-x)^2}$, $|x|<1 \Rightarrow \sum_{n=1}^{\infty} n\cdot x^n = \boxed{\frac{x}{(1-x)^2}}$

c). For $x = \frac{1}{7}$ in b). : $\sum_{n=1}^{\infty} n\left(\frac{1}{7}\right)^n = \frac{1/7}{(1-1/7)^2} = \frac{1}{7}\cdot\frac{1}{\left(\frac{6}{7}\right)^2} = \frac{1}{7}\cdot\frac{49}{36} = \boxed{\frac{7}{36}}$

(8) $f^{(n)}(0) = (n+1)!$

Maclaurin series for f: $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{(n+1)!}{n!} x^n = \boxed{\sum_{n=0}^{\infty} (n+1)x^n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+2)x^{n+1}}{(n+1)x^n}\right| = \frac{n+2}{n+1}\cdot|x| \xrightarrow[n\to\infty]{} |x| < 1 \qquad R=1$$

(9) $f^{(n)}(2) = \frac{(-1)^n \cdot n!}{8^n (n+3)}$

Taylor series for f centered at 2:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot n!}{8^n (n+3) n!}(x-2)^n = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{8^n(n+3)}(x-2)^n}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}(x-2)^{n+1}}{8^{n+1}(n+4)} \cdot \frac{8^n(n+3)}{(-1)^n \cdot (x-2)^n}\right|$$

$$= |x-2| \cdot \frac{n+3}{8(n+4)} \xrightarrow[n\to\infty]{} \frac{|x-2|}{8} < 1 \Rightarrow |x-2| < 8 \qquad R=8$$

(10) a). $f(x) = \sin\left(\frac{\pi x^2}{5}\right) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{\left(\frac{\pi x^2}{5}\right)^{2n+1}}{(2n+1)!} = \boxed{\sum_{n=0}^{\infty} (-1)^n \cdot \frac{\pi^{2n+1}}{5^{2n+1}} \frac{x^{4n+2}}{(2n+1)!}}$

b). $f(x) = e^{-5x} = \sum_{n=0}^{\infty} \frac{(-5x)^n}{n!} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 5^n}{n!} x^n}$

c). $f(x) = 4e^x + e^{6x} = 4\sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{(6x)^n}{n!}$

$$= \sum_{n=0}^{\infty} \frac{4x^n}{n!} + \sum_{n=0}^{\infty} \frac{6^n \cdot x^n}{n!} = \boxed{\sum_{n=0}^{\infty} \frac{4+6^n}{n!} x^n}$$

(11) $$\int \frac{e^x-1}{2x}\,dx = \int \frac{1}{2x}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!} - 1\right)dx$$

$$= \frac{1}{2}\int \frac{1}{x}\left(1+\sum_{n=1}^{\infty}\frac{x^n}{n!} - 1\right)dx$$

$$= \frac{1}{2}\int \frac{1}{x}\sum_{n=1}^{\infty}\frac{x^n}{n!}\,dx = \frac{1}{2}\int \sum_{n=1}^{\infty}\frac{x^{n-1}}{n!}\,dx$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\int \frac{x^{n-1}}{n!}\,dx = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n!}\cdot\frac{x^n}{n} + C$$

$$= \boxed{C + \sum_{n=1}^{\infty}\frac{x^n}{(2n)\cdot n!}}$$

(12) $$\int \frac{\cos(x)-1}{x}\,dx = \int \frac{1}{x}\left(\sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^{2n}}{(2n)!} - 1\right)dx$$

$$= \int \frac{1}{x}\left(1+\sum_{n=1}^{\infty}(-1)^n\cdot\frac{x^{2n}}{(2n)!} - 1\right)dx$$

$$= \int \frac{1}{x}\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}\,dx = \int \sum_{n=1}^{\infty}\frac{x^{2n-1}(-1)^n}{(2n)!}\,dx$$

$$= \sum_{n=1}^{\infty}\int \frac{x^{2n-1}(-1)^n}{(2n)!}\,dx = \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}\cdot\frac{x^{2n}}{2n} + C$$

$$= \boxed{C + \sum_{n=1}^{\infty}(-1)^n\cdot\frac{x^{2n}}{(2n)(2n)!}}$$