M133 - Series: Extra Problems (2)
Sections 11.2 — 11.4 (Solutions)

I. $7+6+\frac{36}{7}+\frac{216}{49}+\dots = 7+7\cdot\frac{6}{7}+7\cdot\frac{36}{49}+7\cdot\frac{216}{7^3}+\dots$

$$= \sum_{n=1}^{\infty} 7\left(\frac{6}{7}\right)^{n-1} = \frac{7}{1-\frac{6}{7}} = \frac{7}{\frac{1}{7}} = \boxed{49}$$

(convergent because the ratio $r=\frac{6}{7}<1$).

II. $\sum_{n=1}^{\infty}(-6)^n\cdot x^n = \sum_{n=1}^{\infty}(-6x)(-6x)^{n-1}$

This is a geometric series with $a=-6x$ and $r=-6x$.
It converges if and only if $|r|<1$, so

$$|-6x|<1 \Rightarrow -1<-6x<1$$

$$\Rightarrow \frac{1}{6}>x>-\frac{1}{6} \Rightarrow \boxed{x\in\left(-\frac{1}{6},\frac{1}{6}\right)}$$

Note: This comes from multiplying every term in $-1<-6x<1$ by $\left(-\frac{1}{6}\right)$. Careful: when multiplying terms in an inequality by a negative number, the directions of the $<, >, \leq, \geq$ signs change!

Note: This comes from:

$$\boxed{\begin{array}{c} |☺|<a \\ \Rightarrow -a<☺<a \end{array}}$$

!

III. Recall that for a series $\sum_{n=1}^{\infty}a_n$:

$$\boxed{\begin{array}{l} a_1=S_1 \\ a_n=S_n-S_{n-1} \text{ for all } n>1 \end{array}}$$

So:

a). $S_1=a_1=\frac{2\cdot1-1}{1+1}=\frac{1}{2}$

For $n>1$:

$$a_n=S_n-S_{n-1}=\frac{2n-1}{n+1}-\frac{2(n-1)-1}{(n-1)+1}=\frac{2n-1}{n+1}-\frac{2n-3}{n}$$

$$=\frac{n(2n-1)-(n+1)(2n-3)}{n(n+1)}=\frac{\cancel{2n^2}-n-\cancel{2n^2}+3n-2n+3}{n(n+1)}$$

$$\Rightarrow \boxed{a_1=\frac{1}{2};\ a_n=\frac{3}{n(n+1)} \text{ for } n>1}$$

b). $\sum_{n=1}^{\infty}a_n=\lim_{n\to\infty}S_n=\lim_{n\to\infty}\frac{2n-1}{n+1}=\boxed{2}$

IV. The function $f(x) = \frac{\tan(2x)}{6x^2+1}$ is not positive on $[1, \infty)$, so the Integral Test does not apply.

V. The function $f(x) = \frac{1}{x \ln x}$ is continuous, positive & decreasing on $[2, \infty)$ so we may apply the Integral Test:

$$\int_1^\infty \frac{1}{x \ln x}\, dx = \ln(\ln x)\Big|_1^\infty = \infty \quad \text{(divergent)}$$

So $\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$ is divergent by the Integral Test.

VI. a) The function $f(x) = e^{-2x} = \frac{1}{e^{2x}}$ is continuous, positive & decreasing on $[1, \infty)$, so we may apply the Integral Test:

$$\int_1^\infty e^{-2x}\, dx = \frac{e^{-2x}}{-2}\Big|_1^\infty = 0 - \frac{e^{-2}}{-2} = \frac{e^{-2}}{2} < \infty \quad \text{(convergent)}$$

So $\sum_{n=1}^{\infty} e^{-2n}$ converges by the Integral Test.

b). Note that $\sum_{n=1}^{\infty} e^{-2n} = \sum_{n=1}^{\infty} (e^{-2})^n = \sum_{n=1}^{\infty} (e^{-2})(e^{-2})^{n-1}$ is a geometric series with ratio $\frac{1}{e^2} < 1$, so it is convergent.

Moreover: $$\sum_{n=1}^{\infty} e^{-2n} = \frac{e^{-2}}{1-e^{-2}} = \frac{1/e^2}{1-1/e^2} = \boxed{\frac{1}{e^2-1}}.$$

VII. For example, the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, and yet $\lim_{n\to\infty} \frac{1}{n} = 0$.

The alternating harmonic series $\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{n}$ is convergent, and $\lim_{n\to\infty} (-1)^{n-1} \cdot \frac{1}{n} = 0$.

This example illustrates that the Test for Divergence fails if $\lim_{n\to\infty} a_n = 0$ i.e. we cannot say whether or not a series converges based on this test if $a_n \to 0$

VIII

1). $\sum_{n=1}^{\infty} (\cos(25))^n = \sum_{n=1}^{\infty} \cos(25)\,(\cos(25))^{n-1}$ convergent geometric series with ratio $r = \cos(25)$, so $|r| < 1$.

$= \dfrac{\cos(25)}{1-\cos(25)}.$

2). $\sum_{n=1}^{\infty} \dfrac{9+7^n}{6^n}$ divergent by the Comparison Test:

$$\frac{9+7^n}{6^n} > \frac{7^n}{6^n}$$

$\sum_{n=1}^{\infty} \dfrac{7^n}{6^n}$ is a divergent geometric series (ratio $r = \frac{7}{6} > 1$).

3). $\sum_{n=1}^{\infty} \arctan(10n)$ divergent by the Test for Divergence:

$$\lim_{n\to\infty} \arctan(10n) = \frac{\pi}{2} \neq 0.$$

4). $\sum_{n=1}^{\infty} \dfrac{2+\sin(n)}{9^n}$ convergent by the Comparison Test:

$$\frac{2+\sin(n)}{9^n} \leq \frac{2+1}{9^n} = \frac{3}{9^n}$$

$\sum_{n=1}^{\infty} \dfrac{3}{9^n} = 3 \cdot \sum_{n=1}^{\infty} \dfrac{1}{9^n}$ is a convergent geometric series (ratio $r = \frac{1}{9} < 1$)

5). $\sum_{n=1}^{\infty} \dfrac{2n^2}{3n^6+2n+1}$ converges by the Limit Comparison Test:

$a_n = \dfrac{2n^2}{3n^6+2n+1}$; $b_n = \dfrac{1}{n^4}$ (convergent p-series)

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2 \cdot n^4}{3n^6+2n+1} = \frac{2}{3}$$

6). $\sum_{n=1}^{\infty} \dfrac{2n^5}{3n^6+2n+1}$ diverges by the Limit Comparison Test:

$a_n = \dfrac{2n^5}{3n^6+2n+1}$; $b_n = \dfrac{1}{n}$ (divergent; harmonic series)

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^5 \cdot n}{3n^6+2n+1} = \frac{2}{3}$$

7). $\frac{1}{4}+\frac{1}{8}+\frac{1}{12}+\frac{1}{16}+\frac{1}{20}+\ldots = \sum_{n=1}^{\infty}\frac{1}{4n} = \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n}$

divergent (constant multiple of the harmonic series).

8). $\sum_{n=1}^{\infty}\frac{1+5^n}{8^n}$ convergent by Comparison Test :

$$\frac{1+5^n}{8^n} < \frac{5^n+5^n}{8^n} = \frac{2\cdot 5^n}{8^n}$$

$\sum_{n=1}^{\infty}\frac{2\cdot 5^n}{8^n} = 2\sum_{n=1}^{\infty}\left(\frac{5}{8}\right)^n$ convergent geometric seri

(ratio $r=\frac{5}{8}<1$)

9). $\sum_{n=1}^{\infty}\ln\left(\frac{n^2+6}{4n^2+3}\right)$ divergent by Test for Divergence :

$$\lim_{n\to\infty}\ln\left(\frac{n^2+6}{4n^2+3}\right) = \ln\left(\frac{1}{4}\right) \neq 0$$

10). $\sum_{n=1}^{\infty}\frac{\sin^2(n)}{n^8+8}$ convergent by Comparison Test :

$$\frac{\sin^2(n)}{n^8+8} \leq \frac{1}{n^8+8} < \frac{1}{n^8}$$

$\sum_{n=1}^{\infty}\frac{1}{n^8}$ convergent p-series.

11). $\sum_{n=1}^{\infty}\frac{\arctan(200n)}{n^{1.001}}$ convergent by the Comparison Test :

$$\frac{\arctan(200n)}{n^{1.001}} \leq \frac{\pi/2}{n^{1.001}}$$

$\sum_{n=1}^{\infty}\frac{\pi/2}{n^{1.001}}$ converges (constant multiple of a convergent p-series)

12). $\sum_{n=1}^{\infty} \frac{6n^2(6n-1)!}{(6n+1)!}$ divergent by the Test for Divergence :

$$\lim_{n\to\infty} \frac{6n^2(6n-1)!}{(6n+1)!} = \lim_{n\to\infty} \frac{6n^2}{6n(6n+1)} = \lim_{n\to\infty} \frac{n}{6n+1} = \frac{1}{6} \neq 0$$

13). $\sum_{n=1}^{\infty} \frac{n}{4n^3+2}$ convergent by the Comparison Test

$$\frac{n}{4n^3+2} < \frac{n}{4n^3} = \frac{1}{4n^2}$$ (convergent p-series)

14). $\sum_{n=1}^{\infty} \frac{n^4}{5n^5-4}$ diverges by the Comparison Test

contrast

$$\frac{n^4}{5n^5-4} > \frac{n^4}{5n^5} = \frac{1}{5n}$$

$\sum_{n=1}^{\infty} \frac{1}{5n} = \frac{1}{5} \sum_{n=1}^{\infty} \frac{1}{n}$ diverges (multiple of the harmonic series).

15). $\sum_{n=1}^{\infty} \frac{n^4}{5n^5+4}$ diverges by the Limit Comparison Test :

Try to use Comparison Test again :

$$a_n = \frac{n^4}{5n^5+4} < \frac{n^4}{5n^5} = \frac{1}{5n} = b_n$$

$\sum_{n=1}^{\infty} \frac{1}{5n}$ diverges

This yields no conclusion about the original series $\sum_{n=1}^{\infty} a_n \ldots$

So try the Limit Comparison Test, with $b_n = \frac{1}{n}$ (divergent series) :

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^4 \cdot n}{5n^5+4} = \frac{1}{5}$$

16). $\sum_{n=1}^{\infty} \frac{5^n}{(-3)^{n-1}} = \sum_{n=1}^{\infty} 5 \cdot \left(-\frac{5}{3}\right)^{n-1}$ divergent geometric series (ratio $r = -\frac{5}{3}$, so $|r| = \frac{5}{3} > 1$).

17). $\sum_{n=1}^{\infty} \frac{7^{n+1}}{6^n - 4}$ divergent by Comparison Test:

$$\frac{7^{n+1}}{6^n - 4} > \frac{7^{n+1}}{6^n}$$

$\sum_{n=1}^{\infty} \frac{7^{n+1}}{6^n}$ is a divergent geometric series $\left(r = \frac{7}{6} > 1\right)$

18). $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n^3}$ convergent p-series $(p = 3 > 1)$

19). $\sum_{n=1}^{\infty} \frac{1}{3n+5}$ diverges by Limit Comparison Test

$a_n = \frac{1}{3n+5}$; $b_n = \frac{1}{n}$ (divergent ; harmonic series)

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{3n+5} = \frac{1}{3}$$

20). $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ convergent p-series $\left(p = \frac{3}{2} > 1\right)$

21). $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n}$ divergent by the Comparison Test

$$\frac{e^{1/n}}{n} \geq \frac{1}{n}$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series)

22). $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ divergent p-series $\left(p = \frac{1}{2} < 1\right)$