7.4 | Partial Fractions:

Rational Function: $f(x) = \frac{P(x)}{Q(x)}$, where P, Q are polynomials.

f = "proper" if $\deg(P) < \deg(Q)$. To integrate proper rational functions f, do the partial fraction decomposition of f:

1). $Q(x) = (a_1x+b_1)\cdot\ldots\cdot(a_nx+b_n)$ **Q = Product of linear factors, none repeated.**

$\Rightarrow$ Partial fraction decomposition: $\boxed{f(x) = \frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \ldots + \frac{A_n}{a_nx+b_n}}$

Example: $\int \frac{x-8}{x^2-7x+10}\,dx$

$$\frac{x-8}{x^2-7x+10} = \frac{x-8}{(x-2)(x-5)} = \frac{A}{x-2} + \frac{B}{x-5}$$

$$\Rightarrow \boxed{x-8 = A(x-5) + B(x-2)} \quad \text{Find A \& B.}$$

Method 1: Give some values to x:

$x=5 \Rightarrow -3 = 3B \Rightarrow \boxed{B=-1}$

$x=2 \Rightarrow -6 = -3A \Rightarrow \boxed{A=2}$

Method 2: Match coefficients of the polynomials:

$x-8 = Ax-5A+Bx-2B$

$x-8 = (A+B)x-(5A+2B)$

$\begin{cases} A+B=1 \\ 5A+2B=8 \end{cases}$ Solve linear system:

$2A+2B=2$

$\ominus \quad 3A=6 \quad \boxed{A=2} \quad \boxed{B=-1}$

$$\Rightarrow \int \frac{x-8}{x^2-7x+10}\,dx = \int\left(\frac{2}{x-2} - \frac{1}{x-5}\right)dx = \boxed{2\ln|x-2| - \ln|x-5| + C}$$

2). Q = Product of linear factors, some of which are repeated:

For every repeated linear factor $(ax+b)^k$ in Q, add: $\boxed{\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \ldots + \frac{A_k}{(ax+b)^k}}$ to the partial frac. decomp.

Example: $\int \frac{x^2+1}{(x+8)(x-2)^2}\,dx = \int\left(\frac{13}{20}\frac{1}{x+8} + \frac{7}{20}\frac{1}{x-2} + \frac{1}{2}\frac{1}{(x-2)^2}\right)dx$

$$= \boxed{\frac{13}{20}\ln|x+8| + \frac{7}{20}\ln|x-2| - \frac{1}{2(x-2)} + C}$$

$$\frac{x^2+1}{(x+8)(x-2)^2} = \frac{A}{x+8} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$$x^2+1 = A(x-2)^2 + B(x+8)(x-2) + C(x+8)$$

$x=-8$: $65 = 100A$ $\boxed{A = 13/20}$; $x=2$: $5 = 10C$ $\boxed{C = 1/2}$

$x=0$: $1 = \frac{13}{5} - 16B + 4 \Rightarrow \boxed{B = 7/20}$

3). Q contains irreducible quadratic factors (none repeated):

For every irreducible factor (ax^2+bx+c) in Q, add: $\boxed{\dfrac{AX+B}{ax^2+bX+C}}$ to the partial frac. decomp.

Example: $\displaystyle\int \frac{5X^2+3X+5}{(X-5)(X^2+4)}\,dX = \int\left(\frac{5}{X-5}+\frac{3}{X^2+4}\right)dX = \boxed{5\ln|X-5| + \frac{3}{2}\arctan\left(\frac{X}{2}\right)+C}$

$$\frac{5X^2+3X+5}{(X-5)(X^2+4)} = \frac{A}{X-5} + \frac{BX+C}{X^2+4}$$

$$5X^2+3X+5 = A(X^2+4)+(BX+C)(X-5)$$

$X=5$: $145 = 29A \Rightarrow \boxed{A=5}$

$X=0$: $5 = 20-5C \Rightarrow \boxed{C=3}$

$X=1$: $13 = 25+(B+3)(-4)$
$= 13-4B \Rightarrow \boxed{B=0}$

4). Q contains repeated irreducible quadratic factors:

For every repeated irreducible quadratic $(ax^2+bx+c)^k$ in Q, add: $\frac{A_1X+B_1}{(ax^2+bx+c)} + \frac{A_2X+B_2}{(ax^2+bx+c)^2} + \ldots + \frac{A_kX+\ldots}{(ax^2+bx\ldots}$ to the partial fraction decomp.

Example: $\displaystyle\int \frac{7X^2+9X+7}{(X^2+1)^2}\,dX$

$$\frac{7X^2+9X+7}{(X^2+1)^2} = \frac{AX+B}{X^2+1} + \frac{CX+D}{(X^2+1)^2}$$

$$\begin{aligned}7X^2+9X+7 &= (AX+B)(X^2+1)+CX+D \\ &= AX^3+AX+BX^2+B+CX+D \\ &= AX^3+BX^2+(A+C)X+(B+D)\end{aligned}$$

$$\begin{cases} A=0 \\ B=7 \\ A+C=9 \Rightarrow C=9 \\ B+D=7 \Rightarrow D=0 \end{cases}$$

$$\Rightarrow \int \frac{7X^2+9X+7}{(X^2+1)^2}\,dX = \int \frac{7}{X^2+1}\,dX + \int \frac{9X}{(X^2+1)^2} = \boxed{7\arctan(X) - \frac{9}{2}\frac{1}{X^2+1}+C}$$