7.2 | TRIGONOMETRIC INTEGRALS

I) Sin & Cos Integrals:

Ia): $\int \sin^m(x)\cos^n(x)\,dx$

If the power of sin / cos is odd,
isolate a factor of sin / cos, and
use $\sin^2+\cos^2=1$ to express all remaining terms as cos / sin

1. $\int \cos^3 x\,dx = \int \cos^2 x\cdot\cos x\,dx$

$= \int (1-\sin^2 x)\cdot\cos x\,dx$ $\quad u=\sin x$, $du=\cos x\,dx$

$= \int (1-u^2)\,du = u-\frac{1}{3}u^3+C$

$= \boxed{\sin x-\frac{1}{3}\sin^3 x+C}$

2. $\int \sin^3(2x)\cos^2(2x)\,dx = \int \sin(2x)\cdot\sin^2(2x)\cos^2(2x)\,dx$

$= \int \sin(2x)\,(1-\cos^2(2x))\cos^2(2x)\,dx$ $\quad u=\cos(2x)$, $du=-2\sin(2x)\,dx$

$= -\frac{1}{2}\int (1-u^2)u^2\,du = -\frac{1}{2}\int (u^2-u^4)\,du$

$= \frac{1}{2}\left(\frac{1}{5}u^5-\frac{1}{3}u^3\right)+C = \boxed{\frac{1}{10}\cos^5(2x)-\frac{1}{6}\cos^3(2x)+C}$

If both powers of sin & cos are even, use:

$$\boxed{\begin{aligned}\sin^2(x)&=\tfrac{1}{2}(1-\cos(2x))\\ \cos^2(x)&=\tfrac{1}{2}(1+\cos(2x))\end{aligned}}$$

$$\boxed{\sin(x)\cos(x)=\tfrac{1}{2}\sin(2x)}$$

(Sometimes also useful)

3. $\int \sin^4 x\,dx = \int (\sin^2 x)^2\,dx = \int \left(\frac{1}{2}(1-\cos(2x))\right)^2 dx$

$= \frac{1}{4}\int (1-2\cos(2x)+\cos^2(2x))\,dx = \frac{1}{4}\left(x-\sin(2x)+\int \underbrace{\cos^2(2x)}_{\frac{1}{2}(1+\cos(4x))}\,dx\right)$

$= \frac{1}{4}\left(x-\sin(2x)+\frac{1}{2}\left(x+\frac{1}{4}\sin(4x)\right)\right)$

$= \boxed{\frac{1}{4}\left(\frac{3}{2}x-\sin(2x)+\frac{1}{8}\sin(4x)\right)+C}$

4. $\int \sin^2(7x)\cos^2(7x)\,dx = \int (\sin(7x)\cos(7x))^2\,dx = \int \left(\frac{1}{2}\sin(14x)\right)^2 dx$

$= \frac{1}{4}\int \sin^2(14x)\,dx = \frac{1}{4}\int \frac{1}{2}(1-\cos(28x))\,dx = \boxed{\frac{1}{8}\left(x-\frac{1}{28}\sin(28x)\right)+C}$

<u>Remark</u>: #4 also works by using the half-angle formulas directly:

4. $\int \sin^2(7x)\cos^2(7x)\,dx = \int \frac{1}{2}(1-\cos(14x))\cdot\frac{1}{2}(1+\cos(14x))\,dx$

$= \frac{1}{4}\int(1-\cos^2(14x))\,dx = \frac{1}{4}\left(x - \int\frac{1}{2}(1+\cos(28x))\,dx\right)$

$= \frac{1}{4}\left(x - \frac{1}{2}x - \frac{1}{2}\cdot\frac{1}{28}\sin(28x)\right) + C = \boxed{\frac{1}{8}\left(x - \frac{1}{28}\sin(28x)\right) + C}$

(IIb): $\boxed{\begin{array}{l}\int \sin(mx)\cos(nx)\,dx \\ \int \sin(mx)\sin(nx)\,dx \\ \int \cos(mx)\cos(nx)\,dx\end{array}}$ use: $\boxed{\begin{array}{l}\sin A\cdot\cos B = \frac{1}{2}[\sin(A-B)+\sin(A+B)] \\ \sin A\cdot\sin B = \frac{1}{2}[\cos(A-B)-\cos(A+B)] \\ \cos A\cdot\cos B = \frac{1}{2}[\cos(A-B)+\cos(A+B)]\end{array}}$

5. $\int \sin(4x)\cos(5x)\,dx = \int \frac{1}{2}\left(\sin(-x)+\sin(9x)\right)dx$

$= \boxed{\frac{1}{2}\left(\cos(x) - \frac{1}{9}\cos(9x)\right) + C}$

An example off-pattern:

6. $\int_{\pi/3}^{\pi/2} \frac{\sin^2 x}{\sqrt{1-\cos x}}\,dx = \int_{\pi/3}^{\pi/2} \frac{\sin^2 x}{\sqrt{1-\cos x}}\,\frac{\sqrt{1+\cos x}}{\sqrt{1+\cos x}}\,dx$

$= \int_{\pi/3}^{\pi/2} \frac{\sin^2 x\sqrt{1+\cos x}}{\underbrace{\sqrt{1-\cos^2 x}}}\,dx$

$\hookrightarrow = \sqrt{\sin^2 x} = |\sin x| = \sin x$ b/c $x \in$ <u>Quad I</u>!

$= \int_{\pi/3}^{\pi/2} \sin x\sqrt{1+\cos x}\,dx = -\frac{2}{3}(1+\cos x)^{3/2}\Big|_{\pi/3}^{\pi/2}$

$= -\frac{2}{3}\left((1)^{3/2} - (1+1/2)^{3/2}\right)$

$= \frac{2}{3}\left(\frac{3}{2}\sqrt{\frac{3}{2}} - 1\right) = \boxed{\sqrt{\frac{3}{2}} - \frac{2}{3}}$

II Sec & Tan Integrals

IIa: $\int \sec^n(x)\tan^m(x)\,dx$

If the power of sec is even, save a factor of $\sec^2$ and use $\boxed{\sec^2 - \tan^2 = 1}$ to express everything else in terms of tan

$\sec^2 x = \frac{d}{dx}(\tan x)$, so this sets up a Substitution, with $u = \tan x$

7. $\int \tan^4 x \sec^4 x\,dx = \int \tan^4 x \cdot \underbrace{\sec^2 x}_{1+\tan^2 x} \cdot \sec^2 x\,dx$

$= \int \tan^4 x\,(1+\tan^2 x)\,\underbrace{\sec^2 x\,dx}_{du}$ $\quad u = \tan x$

$= \int (u^4 + u^6)\,du = \frac{1}{5}u^5 + \frac{1}{7}u^7 + C = \boxed{\frac{1}{5}\tan^5 x + \frac{1}{7}\tan^7 x + C}$

If the power of tan is odd, save a factor of $\sec \cdot \tan$ and use $\boxed{\sec^2 - \tan^2 = 1}$ to express everything else in terms of sec

$\sec x \cdot \tan x = \frac{d}{dx}(\sec x)$, so this sets up a Substitution w/ $u = \sec x$

8. $\int \tan^5 x \sec^6 x\,dx = \int \tan^4 x \cdot \sec^5 x\,(\sec x \tan x)\,dx$

$= \int (\sec^2 x - 1)^2 \sec^5 x\,\underbrace{(\sec x \tan x)\,dx}_{du}$ $\quad u = \sec x$

$= \int (u^4 - 2u^2 + 1)u^5\,du = \frac{1}{10}u^{10} - 2\cdot\frac{1}{8}u^8 + \frac{1}{6}u^6 + C$

$= \boxed{\frac{1}{10}\sec^{10} x - \frac{1}{4}\sec^8 x + \frac{1}{6}\sec^6 x + C}$

(Also works the other way):

8. $\int \tan^5 x \sec^6 x\,dx = \int \tan^5 x \sec^4 x \cdot \sec^2 x\,dx$

$= \int \tan^5 x\,(1+\tan^2 x)^2\,\underbrace{\sec^2 x\,dx}_{du}$ $\quad u = \tan x$

$= \int u^5(1 + 2u^2 + u^4)\,du$

$= \frac{1}{6}u^6 + 2\cdot\frac{1}{8}u^8 + \frac{1}{10}u^{10} + C$

$= \boxed{\frac{1}{6}\tan^6 x + \frac{1}{4}\tan^8 x + \frac{1}{10}\tan^{10} x + C}$

(These answers are really the same)

IIb: Other sec/tan integrals:

$\sec^2 x = 1 + \tan^2 x$

$\frac{d}{dx}\tan x = \sec^2 x$; $\frac{d}{dx}\sec x = \sec x \tan x$

$\int \tan x\,dx = \ln|\sec x| + C$

$\int \sec x\,dx = \ln|\sec x + \tan x| + C$

9. $\int \tan^3 x\,dx$

$= \int \tan x \cdot \tan^2 x\,dx$

$= \int \tan x \cdot (\sec^2 x - 1)\,dx$

$= \int \tan x \cdot \sec^2 x\,dx - \int \tan x\,dx = \boxed{\frac{1}{2}\tan^2 x - \ln|\sec x| + C}$

10. $\int \sec^3 x\,dx =: I$

$= \int \sec x \cdot \underbrace{\sec^2 x}_{(\tan x)'}\,dx$

By Parts: $u = \sec x$ $\quad du = \sec x \tan x\,dx$

$dv = \sec^2 x\,dx$ $\quad v = \tan x$

$= \sec x \cdot \tan x - \int \sec x \cdot \tan^2 x\,dx$

$= \sec x \cdot \tan x - \int \sec x\,(\sec^2 x - 1)\,dx$

$= \sec x \cdot \tan x - \underbrace{\int \sec^3 x\,dx}_{I} + \underbrace{\int \sec x\,dx}_{\ln|\sec x + \tan x|} = I$

$2I = \sec x \cdot \tan x + \ln|\sec x + \tan x|$

$\boxed{I = \frac{1}{2}(\sec x \cdot \tan x + \ln|\sec x + \tan x|)}$