# 7.1 Integration by Parts

$$\boxed{\int f(x)\cdot g'(x)dx = f(x)\cdot g(x) - \int f'(x)\cdot g(x)}$$

$u = f(x) \Rightarrow du = f'(x)dx$
$v = g(x) \Rightarrow dv = g'(x)dx$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

**1.** $\int x^2 \cos x\,dx$ Which function would you rather differentiate later, $x^2$ or $\cos x$? Probably $x^2$ - b/c under differentiation it will become $2x$.

→ So put $u = x^2 \Rightarrow du = 2x\,dx$
$dv = \cos x\,dx \Rightarrow v = \sin x$

$$\begin{aligned}
\int x^2 \cos x\,dx &= \int x^2 (\sin x)'dx \\
&= x^2\cdot \sin x - \int (x^2)' \sin x\,dx \\
&= x^2\cdot \sin x - \underbrace{\int 2x \sin x\,dx}
\end{aligned}$$

Do it again! $u = 2x \Rightarrow du = 2dx$
$dv = \sin x\,dx \Rightarrow v = -\cos x$

$$\begin{aligned}
&= x^2\cdot \sin x - \int 2x\,(-\cos x)'dx \\
&= x^2\cdot \sin x - \left[2x\cdot(-\cos x) - \int (2x)'(-\cos x)dx\right] \\
&= x^2\cdot \sin x + 2x\cos x - \int 2\cos x\,dx \\
&= \boxed{x^2 \sin x + 2x\cos x - 2\sin x + C}
\end{aligned}$$

**2.** $\int e^x \sin(2x)dx =: I$

$$I = \int (e^x)' \sin(2x)dx \qquad \begin{pmatrix} u = \sin(2x) & du = 2\cos(2x) \\ dv = e^x dx & v = e^x \end{pmatrix}$$

$$= e^x \sin(2x) - \int e^x \cdot 2\cos(2x)dx$$

$$= e^x \sin(2x) - 2\int (e^x)'\cos(2x)dx \qquad \begin{pmatrix} u = \cos(2x) & du = -2\sin(2x) \\ dv = e^x dx & v = e^x \end{pmatrix}$$

$$\begin{aligned}
&= e^x \sin(2x) - 2e^x\cos(2x) + 2\int e^x (\cos(2x))'dx \\
&= e^x \sin(2x) - 2e^x\cos(2x) - 2\int e^x\cdot 2\sin(2x)dx \\
&= e^x \sin(2x) - 2e^x\cos(2x) - 4\underbrace{\int e^x \sin(2x)dx}_{I}
\end{aligned}$$

$$\Rightarrow I = e^x\sin(2x) - 2e^x\cos(2x) - 4I$$

$$\Rightarrow 5I = e^x \sin(2x) - 2e^x\cos(2x) \Rightarrow \quad I = \boxed{\frac{1}{5}e^x\left(\sin(2x) - 2\cos(2x)\right) + C}$$

**Remark**: In these problems you can actually start w/ either one of the functions (see next) →

$I := \int e^x \sin(2x)\,dx$ This time put $\begin{pmatrix} u = e^x & du = e^x dx \\ dv = \sin(2x)dx & v = -\frac{1}{2}\cos(2x) \end{pmatrix}$

$= \int e^x \left(-\frac{1}{2}\cos(2x)\right)' dx$

$= -\frac{1}{2}e^x\cos(2x) + \frac{1}{2}\int e^x \cos(2x)\,dx$ $\begin{pmatrix} u = e^x & du = e^x dx \\ dv = \cos(2x)dx & v = \frac{1}{2}\sin(2x) \end{pmatrix}$

$= -\frac{1}{2}e^x\cos(2x) + \frac{1}{2}\int e^x \left(\frac{1}{2}\sin(2x)\right)' dx$

$= -\frac{1}{2}e^x\cos(2x) + \frac{1}{2}e^x \cdot \frac{1}{2}\sin(2x) - \underbrace{\frac{1}{4}\int e^x \sin(2x)\,dx}_{I}$

You just need to be consistent:
(If your first u = exponential, your second u should also be exponential - and same if you start w/ u = trig function. If you switch, you'll just go back to the start & get $I = I$).

$I = \frac{1}{2}e^x\left(\frac{1}{2}\sin(2x) - \cos(2x)\right) - \frac{1}{4}I$

$\frac{5}{4}I = \frac{1}{2}e^x\left(\frac{1}{2}\sin(2x) - \cos(2x)\right)$

$I = \boxed{\frac{1}{5}e^x\left(\sin(2x) - 2\cos(2x)\right) + C}$ ✓

**3.** $\int \ln(3x)\,dx$

$= \int \underbrace{(x)'}_{1} \ln(3x)\,dx$

$= x\ln(3x) - \int x\left(\ln(3x)\right)' dx$

$= x\ln(3x) - \underbrace{\int x \frac{1}{x}dx}_{\int 1 dx = x}$

$= \boxed{x\ln(3x) - x + C}$

$\begin{pmatrix} u = \ln(3x) & du = \frac{1}{x}dx \\ dv = dx & v = x \end{pmatrix}$

**4.** $\int \arctan(4x)\,dx$

$= \int (x)' \arctan(4x)\,dx$

$= x\arctan(4x) - \int x \cdot \left(\arctan(4x)\right)' dx$

$= x\arctan(4x) - \int x \cdot \frac{4}{1+16x^2}dx$

$= \boxed{x\arctan(4x) - \frac{1}{8}\ln(1+16x^2) + C}$

$\begin{pmatrix} u = \arctan(4x) & du = \frac{4}{1+16x^2}dx \\ dv = dx & v = x \end{pmatrix}$

**5.** $\int x \ln x \, dx = \int \left(\frac{1}{2}x^2\right)' \ln x \, dx$

$$\left( \begin{array}{ll} u = \ln x & du = \frac{1}{x}dx \\ dv = x\,dx & v = \frac{1}{2}x^2 dx \end{array} \right)$$

$$= \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x^2 \cdot \frac{1}{x} dx$$

$$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x \, dx$$

$$= \boxed{\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C}$$

**6.** $\int x \cos(2-x)\, dx$

$$\left( \begin{array}{ll} u = x & du = dx \\ dv = \cos(2-x)dx & v = -\sin(2-x) \end{array} \right)$$

$$= \int x \left(-\sin(2-x)\right)' dx$$

$$= -x \sin(2-x) - \int -\sin(2-x)\, dx$$

$$= \boxed{-x \sin(2-x) + \cos(2-x) + C}$$

**7.** $\int x \cdot 2^x dx$

$$\left( \begin{array}{ll} u = x & du = dx \\ dv = 2^x dx & v = 2^x \frac{1}{\ln 2} \end{array} \right)$$

$$= x \cdot 2^x \frac{1}{\ln 2} - \int 2^x \frac{1}{\ln 2} dx$$

$$= \boxed{x \cdot 2^x \frac{1}{\ln 2} - 2^x \frac{1}{(\ln 2)^2} + C}$$

**8.** $\int (\ln x)^2 dx$

$$\left( \begin{array}{ll} u = (\ln x)^2 & du = 2\ln x \cdot \frac{1}{x} \\ dv = dx & v = x \end{array} \right)$$

$$= \int (x)' \cdot (\ln x)^2 dx$$

$$= x \cdot (\ln x)^2 - \int x \cdot 2 \ln x \cdot \frac{1}{x} dx$$

$$= x \cdot (\ln x)^2 - 2 \underbrace{\int \ln x \, dx}_{= x \ln x - x}$$

(Class example; just like #3)

$$= \boxed{x \cdot (\ln x)^2 - 2x \ln x + 2x + C}$$

9. $\int \frac{\ln x}{x^2}dx$

$$\left(\begin{array}{ll} u=\ln x & du=\frac{1}{x}dx \\ dv=\frac{1}{x^2}dx & v=\frac{-1}{x} \end{array}\right)$$

$$= -\frac{1}{x}\ln x - \int \frac{-1}{x}\frac{1}{x}dx$$

$$= \boxed{-\frac{1}{x}\ln x - \frac{1}{x} + C}$$

10. $\int e^{\sqrt{x}}dx$

Use Substitution First: $\boxed{\sqrt{x}=y}$

$$\frac{1}{2\sqrt{x}}dx = dy \Rightarrow dx = 2\underbrace{\sqrt{x}}_{y}dy = 2y\,dy \quad \boxed{dx=2y\,dy}$$

$$\Rightarrow \int e^{\sqrt{x}}dx = \int e^y \cdot 2y\,dy \longleftarrow \text{By Parts}$$

$$= 2\int e^y \cdot y\,dy \qquad \left(\begin{array}{ll} u=y & du=dy \\ dv=e^y dy & v=e^y \end{array}\right)$$

$$= 2\left(ye^y - \int e^y dy\right)$$

$$= 2\left(ye^y - e^y\right) + C$$

$$= \boxed{2\left(\sqrt{x}e^{\sqrt{x}} - e^{\sqrt{x}}\right) + C}$$

11. $\int_1^4 \sqrt{x}\ln x\,dx$

$$\left(\begin{array}{ll} u=\ln x & du=\frac{1}{x}dx \\ dv=\sqrt{x}dx & v=\frac{2}{3}x^{3/2} \end{array}\right)$$

$$= \ln x \cdot \frac{2}{3}x^{3/2}\Big|_1^4 - \int_1^4 \frac{2}{3}x^{3/2}\frac{1}{x}dx$$

$$= \ln(4)\cdot\frac{2}{3}\cdot 4^{3/2} - \underbrace{\ln(1)}_{0}\cdot\frac{2}{3}1^{3/2} - \frac{2}{3}\int_1^4 x^{1/2}dx$$

$$= \frac{16}{3}\ln(4) - \frac{2}{3}\,\frac{2}{3}x^{3/2}\Big|_1^4$$

$$= \frac{16}{3}\ln(4) - \frac{4}{9}(8-1) = \boxed{\frac{16}{3}\ln(4) - \frac{28}{9}}$$

12. $\int_0^2 x e^{9x} dx$ $\quad \left( \begin{array}{ll} u = x & du = dx \\ dv = e^{9x}dx & v = \frac{1}{9}e^{9x} \end{array} \right)$

$$= \frac{x}{9}e^{9x}\Big|_0^2 - \int_0^2 \frac{1}{9}e^{9x}dx = \frac{2}{9}e^{18} - \frac{1}{81}e^{9x}\Big|_0^2 = \frac{2}{9}e^{18} - \frac{1}{81}(e^{18}-1) = \boxed{\frac{17}{81}e^{18} + \frac{1}{81}}$$

13. $\int_1^4 \ln(\sqrt{x})dx$ $\quad \left( \begin{array}{ll} u = \ln(\sqrt{x}) & du = \frac{1}{\sqrt{x}}\frac{1}{2\sqrt{x}}dx = \frac{1}{2x}dx \\ dv = dx & v = x \end{array} \right)$

$$= x\ln(\sqrt{x})\Big|_1^4 - \int_1^4 \cancel{x}\frac{1}{2\cancel{x}}dx = 4\ln 2 - \frac{1}{2}x\Big|_1^4 = 4\ln 2 - 2 + \frac{1}{2} = \boxed{4\ln(2) - \frac{3}{2}}$$

14. $\int \cos x \ln(\sin x)dx$

Substitution: $\left. \begin{array}{l} y = \sin x \\ dy = \cos x\,dx \end{array} \right\} \Rightarrow \int \cos x \ln(\sin x)dx = \int \ln(y)dy =$ (By Parts):

$$= y\ln(y) - y + C$$
$$= \boxed{\sin x \cdot \ln(\sin x) - \sin x + C}$$

15. $\int_1^2 x^4(\ln x)^2 dx$ $\quad \left( \begin{array}{ll} u = (\ln x)^2 & du = 2\ln x \cdot \frac{1}{x} \\ dv = x^4dx & v = \frac{1}{5}x^5 \end{array} \right)$

$$= \frac{1}{5}x^5(\ln x)^2\Big|_1^2 - \int_1^2 \frac{1}{5}x^5 \cdot 2\ln x \cdot \frac{1}{x}dx$$

$$= \frac{32}{5}(\ln 2)^2 - \frac{2}{5}\int_1^2 x^4 \ln x\,dx \quad \left( \begin{array}{ll} u = \ln x & du = \frac{1}{x}dx \\ dv = x^4dx & v = \frac{1}{5}x^5 \end{array} \right)$$

$$= \frac{32}{5}(\ln 2)^2 - \frac{2}{5}\left( \frac{1}{5}x^5\ln x\Big|_1^2 - \int_1^2 \frac{1}{5}x^5\frac{1}{x}dx \right) = \frac{32}{5}(\ln 2)^2 - \frac{2}{5}\left( \frac{32}{5}\ln 2 - \frac{1}{5}\frac{1}{5}x^5\Big|_1^2 \right)$$

$$= \frac{32}{5}(\ln 2)^2 - \frac{64}{25}\ln 2 + \frac{2}{125}(32-1) = \boxed{\frac{32}{5}(\ln 2)^2 - \frac{64}{25}\ln 2 + \frac{62}{125}}$$

16. $\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3 \cos(\theta^2)d\theta$

Substitution: $x = \theta^2$ $\qquad \theta = \sqrt{\pi/2} \Rightarrow x = \pi/2$

$dx = 2\theta d\theta \Rightarrow d\theta = \frac{1}{2\theta}dx \qquad \theta = \sqrt{\pi} \Rightarrow x = \pi$

$$= \int_{\pi/2}^{\pi} \underbrace{\theta^3\cos(\theta^2)\frac{1}{2\theta}}_{\frac{1}{2}\theta^2\cos(\theta^2) = \frac{1}{2}x\cos x}dx = \frac{1}{2}\int_{\pi/2}^{\pi} x\cos(x)dx$$

By Parts $\left( \begin{array}{ll} u = x & du = dx \\ dv = \cos x\,dx & v = \sin x \end{array} \right)$

$$= \frac{1}{2}\left( x\sin x\Big|_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi}\sin x\,dx \right)$$

$$= \frac{1}{2}\left( \pi\sin\pi - \frac{\pi}{2}\sin\frac{\pi}{2} + \cos x\Big|_{\pi/2}^{\pi} \right)$$

$$= \frac{1}{2}\left( -\frac{\pi}{2} + \cos\pi - \cos\frac{\pi}{2} \right) = \frac{1}{2}\left( -\frac{\pi}{2} - 1 \right) = \boxed{-\frac{1}{2} - \frac{\pi}{4}}$$