L'Hospital's Rule: Summary

A. The Basic Indeterminate Forms $\boxed{\frac{0}{0}}$ and $\boxed{\frac{\infty}{\infty}}$

L'Hospital's Rule: If f and g are differentiable, and $g'(x) \neq 0$ on an open interval I containing a (though not necessarily at a), and if

$$\boxed{\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0} \quad \text{OR} \quad \boxed{\lim_{x\to a} f(x) = \pm\infty \ \& \ \lim_{x\to a} g(x) = \pm\infty}$$

then:

$$\boxed{\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}}$$

↳ Also works for one-sided limits & limits at $\pm\infty$
(i.e. you can take a above to be instead a_+, a_-, or $\pm\infty$).

1. $\displaystyle \lim_{x\to 1} \frac{\ln x}{x-1} \overset{l'H}{\underset{0/0}{=}} \lim_{x\to 1} \frac{\frac{1}{x}}{1} = \lim_{x\to 1} \frac{1}{x} = 1$

2. $\displaystyle \lim_{x\to \infty} \frac{\ln x}{x-1} \overset{l'H}{\underset{\infty/\infty}{=}} \lim_{x\to \infty} \frac{\frac{1}{x}}{1} = \lim_{x\to \infty} \frac{1}{x} = 0$

3. $\displaystyle \lim_{x\to \infty} \frac{e^x}{x^2} \overset{l'H}{\underset{\infty/\infty}{=}} \lim_{x\to \infty} \frac{e^x}{2x} \overset{l'H}{\underset{\infty/\infty}{=}} \lim_{x\to \infty} \frac{e^x}{2} = \infty$

B. Indeterminate Products: $\boxed{0\cdot\infty}$

Careful: $\underbrace{0\cdot 0}_{=0}$ and $\underbrace{\infty\cdot\infty}_{=\infty}$ are not indetermin

Turn them into $\frac{0}{0}$ or $\frac{\infty}{\infty}$:

4. $\displaystyle \lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \overset{l'H}{\underset{\infty/\infty}{=}} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} (-x) = 0$

↳ also could have written: $\displaystyle = \lim_{x\to 0^+} \frac{x}{\frac{1}{\ln x}} \overset{l'H}{\underset{0/0}{=}} \lim_{x\to 0^+} \frac{1}{-\frac{1}{\ln^2 x}\cdot\frac{1}{x}}$

$\displaystyle = \lim_{x\to 0^+} (-x\ln^2 x)$

This is worse than what we started w

C. Indeterminate Difference: $\infty - \infty$

Careful: $\infty + \infty = \infty$ is OK

Usually convert the difference into a quotient ($\frac{0}{0}$ or $\frac{\infty}{\infty}$):

5. $\lim_{x\to 0}\left(\frac{1}{\ln(1+x)} - \frac{1}{x}\right) = \lim_{x\to 0}\frac{x - \ln(1+x)}{x\ln(1+x)} \overset{l'H}{\underset{0/0}{=}} \lim_{x\to 0}\frac{1 - \frac{1}{1+x}}{\ln(1+x) + \frac{x}{1+x}}$

$= \lim_{x\to 0}\frac{\frac{x+1-1}{1+x}}{\frac{(1+x)\ln(1+x)+x}{1+x}} = \lim_{x\to 0}\frac{x}{(1+x)\ln(1+x)+x}$

$\overset{l'H}{\underset{0/0}{=}} \lim_{x\to 0}\frac{1}{\ln(1+x) + (1+x)\frac{1}{1+x} + 1} = \lim_{x\to 0}\frac{1}{\ln(1+x)+2} = \frac{1}{2}$

D. Indeterminate Powers: 0^0 ∞^0 1^∞

Such a limit can be turned into $\frac{0}{0}$ or $\frac{\infty}{\infty}$ using logarithms: $L := \lim_{x\to a}\left([f(x)]^{g(x)}\right)$

$\Rightarrow \ln(L) = \lim_{x\to a}\left(\underbrace{g(x)\cdot\ln[f(x)]}_{\text{usually } 0\cdot\infty}\right)$

6. $\lim_{x\to\infty}(2x)^{\frac{1}{7x}} = 1$ $\quad \infty^0$

$L := \lim_{x\to\infty}(2x)^{\frac{1}{7x}} \Rightarrow \ln L = \lim_{x\to\infty}\frac{1}{7x}\ln(2x) = \lim_{x\to\infty}\frac{\ln(2x)}{7x} \overset{l'H}{\underset{\infty/\infty}{=}} \lim_{x\to\infty}\frac{\frac{1}{x}}{7} = 0$

$\ln L = 0 \Rightarrow \boxed{L=1}$

7. $\lim_{x\to 0^+} x^x = 1$ $\quad 0^0$

$L := \lim_{x\to 0^+} x^x \Rightarrow \ln L = \lim_{x\to 0^+} x\ln x = 0$ (Example 4) $\Rightarrow \boxed{L=1}$

8. $\lim_{x\to 0_+}(1+\sin(4x))^{\cot(x)} = e^4$ $\quad 1^\infty$

$\ln L = \lim_{x\to 0_+}\cot(x)\cdot\ln(1+\sin(4x)) = \lim_{x\to 0_+}\frac{\cos x\cdot\ln(1+\sin(4x))}{\sin x}$

$\overset{l'H}{\underset{0/0}{=}} \lim_{x\to 0_+}\frac{-\sin x\cdot\ln(1+\sin(4x)) + \cos x\cdot\frac{4\cos(4x)}{1+\sin(4x)}}{\cos x} = 4$

$\ln L = 4 \Rightarrow \boxed{L = e^4}$