L'Hospital Problems

① $\displaystyle \lim_{x\to\pi/2} \frac{\cos^2 x}{\sin x - 1} \underset{0/0}{\overset{l'H}{=}} \lim_{x\to\pi/2} \frac{-2\cos x \sin x}{\cos x} = \lim_{x\to\pi/2}(-2\sin x) = -2$

② $\displaystyle \lim_{x\to 0} \frac{e^x - x - 1}{\cos x - 1} \underset{0/0}{\overset{l'H}{=}} \lim_{x\to 0} \frac{e^x - 1}{-\sin x} \underset{0/0}{\overset{l'H}{=}} \lim_{x\to 0} \frac{e^x}{-\cos x} = -1$

③ $\displaystyle \lim_{x\to\infty} \frac{\sqrt{x}}{(\ln x)^2} \underset{\infty/\infty}{\overset{l'H}{=}} \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{2\ln x \cdot \frac{1}{x}} = \lim_{x\to\infty} \frac{\sqrt{x}}{4\ln x} \underset{\infty/\infty}{\overset{l'H}{=}} \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{4\frac{1}{x}} = \lim_{x\to\infty} \frac{\sqrt{x}}{8} = \infty$

④ $\displaystyle \lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x\to 0} \frac{x - \sin x}{x \sin x} \underset{0/0}{\overset{l'H}{=}} \lim_{x\to 0} \frac{1 - \cos x}{\sin x + x\cos x} =$

$\infty - \infty$

$\displaystyle \underset{0/0}{\overset{l'H}{=}} \lim_{x\to 0} \frac{\sin x}{\cos x + \cos x - x\sin x} = \frac{0}{2} = 0$

⑤ $\displaystyle \lim_{x\to 0} (\sin x)(\ln x) = \lim_{x\to 0} \frac{\ln x}{\frac{1}{\sin x}} \underset{\infty/\infty}{\overset{l'H}{=}} \lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{\cos x}{\sin^2 x}} = \lim_{x\to 0} \frac{-\sin^2 x}{x\cos x}$

$0 \cdot \infty$

$\displaystyle \underset{0/0}{\overset{l'H}{=}} \lim_{x\to 0} \frac{-2\sin x \cos x}{\cos x - x\sin x} = \frac{0}{1} = 0.$

also try: $\displaystyle = \lim_{x\to 0} \frac{\sin x}{\frac{1}{\ln x}} \underset{0/0}{\overset{l'H}{=}} \lim_{x\to 0} \frac{\cos x}{\frac{-1}{\ln^2 x} \cdot \frac{1}{x}} = \lim_{x\to 0}\left(-x \cdot \ln^2 x \cdot \cos x\right)$

$(0 \cdot \infty)$ worse than original

⑥ $\displaystyle \lim_{x\to 1} \frac{e^x - e}{\ln x} \underset{0/0}{\overset{l'H}{=}} \lim_{x\to 1} \frac{e^x}{\frac{1}{x}} = \lim_{x\to 1} x e^x = e$

⑦ $\displaystyle \lim_{x\to\infty} x^{1/x} = 1.$ $\quad \infty^0$

$\displaystyle L = \lim_{x\to\infty} x^{1/x} \Rightarrow \ln L = \lim_{x\to\infty} \frac{1}{x}\ln x \underset{\infty/\infty}{\overset{l'H}{=}} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0 \Rightarrow \boxed{L = 1}$

⑧ $\displaystyle \lim_{x\to\infty} \frac{3x - 1}{7 - 12x} \underset{\infty/\infty}{\overset{l'H}{=}} \lim_{x\to\infty} \frac{3}{-12} = -\frac{1}{4}.$

⑨ $\displaystyle \lim_{x\to\infty} \frac{x^2}{e^x} \underset{\infty/\infty}{\overset{l'H}{=}} \lim_{x\to\infty} \frac{2x}{e^x} \underset{\infty/\infty}{\overset{l'H}{=}} \lim_{x\to\infty} \frac{2}{e^x} = 0.$

(10) $\lim\limits_{x\to 0} \dfrac{\cos x - \cos^2 x}{\sin x} \overset{l'H}{\underset{0/0}{=}} \lim\limits_{x\to 0} \dfrac{-\sin x + 2\cos x \sin x}{\cos x} = \dfrac{0}{1} = 0.$

(11) $\lim\limits_{x\to 0} \dfrac{\sin 4x}{\sin 3x} \overset{l'H}{\underset{0/0}{=}} \lim\limits_{x\to 0} \dfrac{4\cos 4x}{3\cos 3x} = \dfrac{4}{3}$

(12) $\lim\limits_{x\to 0} \left(\cot x - \dfrac{1}{x}\right) = \lim\limits_{x\to 0} \left(\dfrac{\cos x}{\sin x} - \dfrac{1}{x}\right) = \lim\limits_{x\to 0} \dfrac{x\cos x - \sin x}{x \sin x}$

$\boxed{\infty - \infty}$

$\overset{l'H}{\underset{0/0}{=}} \lim\limits_{x\to 0} \dfrac{\cancel{\cos x} - x\sin x - \cancel{\cos x}}{\sin x + x\cos x} = \lim\limits_{x\to 0} \dfrac{-x\sin x}{\sin x + x\cos x}$

$\overset{l'H}{\underset{0/0}{=}} \lim\limits_{x\to 0} \dfrac{-\sin x - x\cos x}{\cos x + \cos x - x\sin x} = \dfrac{0}{2} = 0.$

(13) $\lim\limits_{x\to 1} \dfrac{x(\ln x - 1) + 1}{(x-1)\ln x} \overset{l'H}{\underset{0/0}{=}} \lim\limits_{x\to 1} \dfrac{(\ln x - 1) + x\cdot\frac{1}{x}}{\ln x + \frac{x-1}{x}} = \lim\limits_{x\to 1} \dfrac{x\ln x}{x\ln x + x - 1}$

$\overset{l'H}{\underset{0/0}{=}} \lim\limits_{x\to 1} \dfrac{\ln x + x\cdot\frac{1}{x}}{\ln x + 1 + 1} = \dfrac{1}{2}$

(14) $\lim\limits_{x\to 0} \dfrac{\sin x - x\cos x}{x - \sin x} \overset{l'H}{\underset{0/0}{=}} \lim\limits_{x\to 0} \dfrac{\cancel{\cos x} - \cancel{\cos x} + x\sin x}{1 - \cos x}$

$\overset{l'H}{\underset{0/0}{=}} \lim\limits_{x\to 0} \dfrac{\sin x + x\cos x}{\sin x}$

$\overset{l'H}{\underset{0/0}{=}} \lim\limits_{x\to 0} \dfrac{\cos x + \cos x - x\sin x}{\cos x} = 2$

(15) $\lim\limits_{x\to 1} \dfrac{1 - x + \ln x}{1 + \cos \pi x} \overset{l'H}{\underset{0/0}{=}} \lim\limits_{x\to 1} \dfrac{-1 + \frac{1}{x}}{-\pi\sin\pi x} \overset{l'H}{\underset{0/0}{=}} \lim\limits_{x\to 1} \dfrac{-\frac{1}{x^2}}{-\pi^2\cos\pi x} = \dfrac{-1}{\pi^2}$

(16) $$\lim_{x\to\infty} x^3 e^{-x^2} = \lim_{x\to\infty} \frac{x^3}{e^{x^2}} \underset{\infty/\infty}{\overset{l'H}{=}} \lim_{x\to\infty} \frac{3x^2}{2x e^{x^2}} = \lim_{x\to\infty} \frac{3x}{2e^{x^2}} \underset{\infty/\infty}{\overset{l'H}{=}} \lim_{x\to\infty} \frac{3}{4x e^{x^2}} =$$

$\boxed{\infty \cdot 0}$

(17) $$\lim_{x\to 0_+} (\cos x)^{1/x^2} = \frac{1}{\sqrt{e}} \qquad 1^\infty$$

$$\ln L = \lim_{x\to 0_+} \frac{1}{x^2} \ln(\cos x) = \lim_{x\to 0_+} \frac{\ln(\cos x)}{x^2} \underset{0/0}{\overset{l'H}{=}} \lim_{x\to 0_+} \frac{\frac{-\sin x}{\cos x}}{2x}$$

$$= \lim_{x\to 0_+} \frac{-\sin x}{2x\cos x} \underset{0/0}{\overset{l'H}{=}} \lim_{x\to 0_+} \frac{-\cos x}{2\cos x - 2x\sin x} = \frac{-1}{2} \Rightarrow \boxed{L = e^{-1/2}}$$

(18) $$\lim_{x\to 1_+} \ln x \cdot \tan\left(\frac{\pi x}{2}\right) = \lim_{x\to 1_+} \frac{\ln x}{\frac{1}{\tan(\pi x/2)}} \underset{0/0}{\overset{l'H}{=}} \lim_{x\to 1_+} \frac{\frac{1}{x}}{\frac{-1}{\tan^2(\pi x/2)} \cdot \sec^2(\pi x/2}$$

$\boxed{0 \cdot \infty}$

$$= \lim_{x\to 1_+} \left(- \frac{2\tan^2(\pi x/2)}{\pi x \sec^2(\pi x/2)} \right) \cancel{\underset{\infty/\infty}{\overset{l'H}{=}} \lim_{x\to 1_+} \left(- \frac{4\tan(\pi x/2) \cdot \frac{\pi}{2} \cdot \sec^2(\pi x/2)}{\pi \sec^2(\pi x/2) + 2\pi x \sec^2(\pi x/2) \cdot \tan(\pi} \right.}$$

still ∞/∞, looks bad...

$$= \lim_{x\to 1_+} \left(- \frac{2\sin^2(\pi x/2)}{\pi x} \right)$$

$$= -\frac{2}{\pi}$$

(19) $$\lim_{x\to 0} (1-2x)^{1/x} = \frac{1}{e^2} \qquad 1^\infty$$

$$\ln L = \lim_{x\to 0} \frac{1}{x} \ln(1-2x) = \lim_{x\to 0} \frac{\ln(1-2x)}{x} \underset{0/0}{\overset{l'H}{=}} \lim_{x\to 0} \frac{\frac{-2}{1-2x}}{1} = -2 \Rightarrow \boxed{L = e^{-2}}$$

(20) $$\lim_{x\to 0_+} \left(\frac{1}{x} - \frac{1}{e^x - 1} \right) = \lim_{x\to 0_+} \frac{e^x - 1 - x}{x(e^x - 1)} \underset{0/0}{\overset{l'H}{=}} \lim_{x\to 0_+} \frac{e^x - 1}{e^x - 1 + xe^x}$$

$\boxed{\infty - \infty}$

$$\underset{0/0}{\overset{l'H}{=}} \lim_{x\to 0_+} \frac{e^x}{e^x + e^x + xe^x} = \frac{1}{2}$$

(21) $\lim_{x\to 0+} x^{\sqrt{x}} = 1$ $\quad 0^0$

$$\ln L = \lim_{x\to 0+} \underbrace{\sqrt{x}\,\ln x}_{0\cdot\infty} = \lim_{x\to 0+} \frac{\ln x}{\frac{1}{\sqrt{x}}} \underset{\infty/\infty}{\overset{l'H}{=}} \lim_{x\to 0+} \frac{\frac{1}{x}}{-\frac{1}{x}\,\frac{1}{2\sqrt{x}}} = \lim_{x\to 0+}\left(-2\sqrt{x}\right) = 0$$

$$\Rightarrow \boxed{L=1}$$

(22) $\lim_{x\to\infty}(x - \ln x) = \lim_{x\to\infty} x\,\underbrace{\left(1 - \frac{\ln x}{x}\right)}_{\to 1 \text{ as } x\to\infty} = \infty$

$\infty - \infty$

$$\lim_{x\to\infty} \frac{\ln x}{x} \underset{\infty/\infty}{\overset{l'H}{=}} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

(23) $\lim_{x\to 1+} x^{1/(1-x)}$ $\quad 1^\infty$

$$\ln L = \lim_{x\to 1+} \frac{\ln x}{1-x} \underset{0/0}{\overset{l'H}{=}} \lim_{x\to 1+} \frac{\frac{1}{x}}{-1} = -1 \Rightarrow \boxed{L = \frac{1}{e}}$$

(24) $$\lim_{x\to 0} \frac{\cos x - 1 + \frac{1}{2}x^2}{x^4} \underset{0/0}{\overset{l'H}{=}} \lim_{x\to 0} \frac{-\sin x + x}{4x^3} \underset{0/0}{\overset{l'H}{=}} \lim_{x\to 0} \frac{-\cos x + 1}{12x^2}$$

$$\underset{0/0}{\overset{l'H}{=}} \lim_{x\to 0} \frac{\sin x}{24x}$$

$$\underset{0/0}{\overset{l'H}{=}} \lim_{x\to 0} \frac{\cos x}{24} = \frac{1}{24}$$

(25) $$\lim_{x\to\infty} x \sin\left(\frac{\pi}{x}\right) = \lim_{x\to\infty} \frac{\sin\left(\frac{\pi}{x}\right)}{\frac{1}{x}} \underset{0/0}{\overset{l'H}{=}} \lim_{x\to\infty} \frac{\cos\left(\frac{\pi}{x}\right)\cdot\frac{-\pi}{x^2}}{-\frac{1}{x^2}}$$

$\infty \cdot 0$

$$= \lim_{x\to\infty}\left(+\pi\cos\left(\frac{\pi}{x}\right)\right) = +\pi\cos(0) = +\pi$$