M133-S60: 9/15/2017

6.5 Exponential Growth and Decay

1. A population P of insects increases at a rate proportional to the current population. Suppose there are 105 insects at time $t = 0$, and 575 insects seven days later.
a). Find an expression for the number $P(t)$ of insects at time $t > 0$, where t is in days.

b). How many insects will there be in 51 days?

2. Suppose that a particular carbon isotope which is found in living trees has a half-life of 5700 years. Scientists have discovered that the charcoal from a tree burned in a volcanic eruption contains 36 percent of the isotope level that is normally found in living trees. How many years ago did the volcanic eruption occur?

3. Suppose you have just placed d dollars in a bank account that pays 5 percent (annual) interest, compounded continuously.
a). How much do you have in the account after 7 years?

b). How long (in years) will it take your money to double?

4. Suppose that the temperature of a cup of soup obeys Newton's law of cooling. If the soup has a temperature of 175°F when served to a customer, and 5 minutes later has cooled to 145°F in a room at 64°F, how much longer must it take the soup to reach a temperature of 110°F? (how many additional minutes)

If the same cup of 175°F soup is instead placed into a freezer set at 10°F, what is the time required for the soup to cool from 175°F to 110°F in this situation?

5. Suppose that the temperature of a loaf of baked bread satisfies Newton's law of cooling. Right now the temperature of the bread is 65°C above room temperature but twenty-five minutes ago, it was 70°C above room temperature.

a). How far above room temperature will the bread be thirty-five minutes from now?

b). How much greater than room temperature will the bread be 4 hours from now?

c). In how many minutes (from now) will the temperature of the bread measure 32°C above room temperature?

Summary of formulas:

Natural growth/decay: If a quantity grows or decays at a rate proportional to its current size, meaning:

$$\frac{dy}{dx} = ky;$$

(growth if $k > 0$, decay if $k < 0$), then

$$\boxed{y(t) = y(0)e^{kt}.}$$

Notable special cases:

1. **Radioactive Decay:**

$$\begin{aligned} m_0 &= \text{Initial mass of some radioactive material;} \\ m(t) &= \text{mass remaining after time } t. \end{aligned}$$

$$\boxed{m(t) = m_0 e^{kt}.}$$

"Half-life" = time required for half of a given quantity of the material to decay.

2. **Continuously Compounded Interest:**

$$\begin{aligned} A_0 &= \text{Initial amount of money invested;} \\ r &= \text{Interest rate;} \\ A(t) &= \text{Amount of money after } t \text{ years if interest is compounded continuously.} \end{aligned}$$

$$\boxed{A(t) = A_0 e^{rt}.}$$

3. **Newton's Law of Cooling:** Rate of cooling of an object is proportional to the temperature difference between the object and its surroundings.

$$\begin{aligned} T_S &= \text{Temperature of Surroundings;} \\ T(t) &= \text{Temperature of object at time } t. \end{aligned}$$

Then $T(t)$ satisfies the ODE:

$$\frac{dT}{dt} = k(T - T_S).$$

To solve, do a change of variable:

$$y(t) = T(t) - T_S.$$

Then y satisfies the ODE:

$$\frac{dy}{dt} = ky.$$

Solution for y:

$$\boxed{y(t) = y(0)e^{kt}.}$$

Solution for T:

$$\boxed{T(t) = T_S + (T(0) - T_S)e^{kt}.}$$