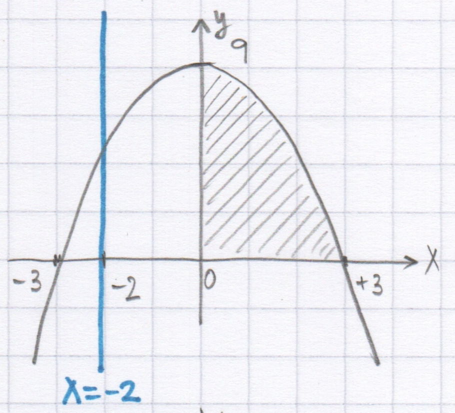
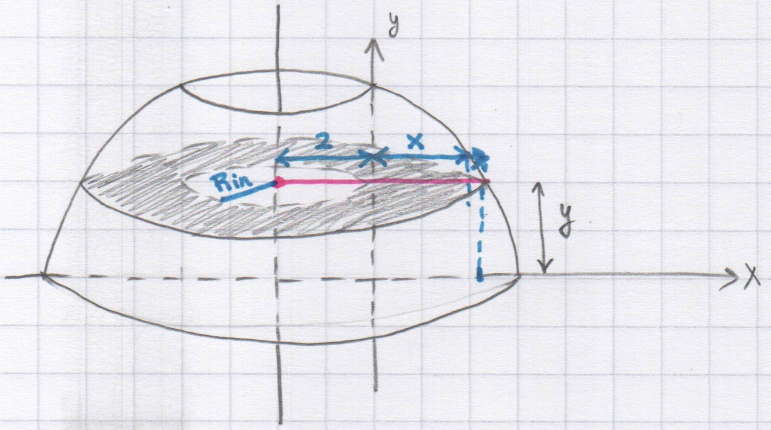
5.2 Volumes (Extra Problems).

① Rotate region under $f(x) = 9 - x^2$, $0 \le x \le 3$, about $x = -2$.





Vertical axis ⇒ want to integrate dy

Cross-sectional area: $R_{out} = 2 + x = 2 + \sqrt{9-y}$

$R_{in} = 2$.

Solve for x: $y = 9 - x^2 \Rightarrow x^2 = 9 - y \Rightarrow \underbrace{x = \sqrt{9-y}}_{\text{Why not } \pm\text{?!}}$

Because $\underline{x \ge 0}$!

$$A(y) = \pi\left(2 + \sqrt{9-y}\right)^2 - \pi \cdot 4$$

$$= \pi\left(\cancel{4} + 4\sqrt{9-y} + (9-y) - \cancel{4}\right)$$

$$Vol = \int_0^9 \pi\left(4\sqrt{9-y} + 9 - y\right)dy = \pi \cdot 4 \int_0^9 \sqrt{9-y}\,dy + \pi\int_0^9 (9-y)\,dy$$

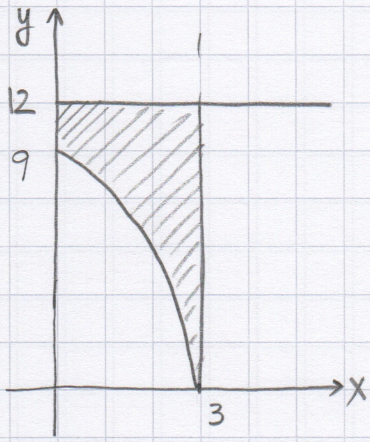
$$= 4\pi\left(-\frac{2}{3}(9-y)^{3/2}\right)\Big|_0^9 + \pi\left(9y - y^2/2\right)\Big|_0^9$$

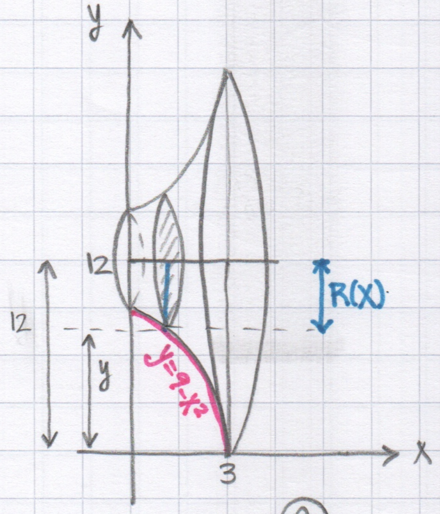
$$= 4\pi\left(0 + \frac{2}{3} \cdot 9^{3/2}\right) + \pi\left(81 - 81/2\right)$$

$\frac{2}{3} \cdot 3^3 = 18$

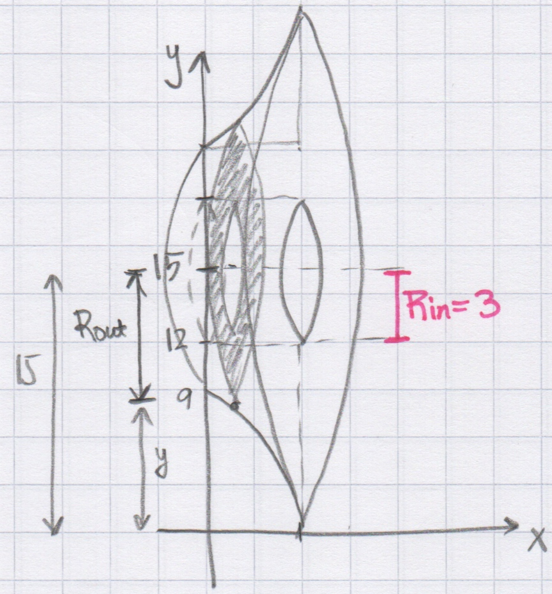
$$= 4\pi \cdot 18 + \pi \cdot \frac{81}{2} = \boxed{\frac{225\pi}{2}}$$

② Region b/w $f(x) = 9 - x^2$, line $y = 12$, for $0 \le x \le 3$, about
(a). $y = 12$. (b). $y = 15$.





ⓐ



Horizontal axis ⇒ want to integrate dx

ⓐ Cross-sections are <u>disks</u>. Radius $R(x) = 12 - y = 12 - (9 - x^2) = \underline{\underline{3 + x^2}}$

$A(x) = \pi\,(3 + x^2)^2$

Solve for y: $y = 9 - x^2$

$$\text{Vol} = \int_0^3 \pi\,(9 + 6x^2 + x^4)\,dx = \pi\left(9x + 2x^3 + \frac{x^5}{5}\right)\Big|_0^3 = \frac{648}{5}\pi.$$

ⓑ Cross-sections are <u>washers</u>.

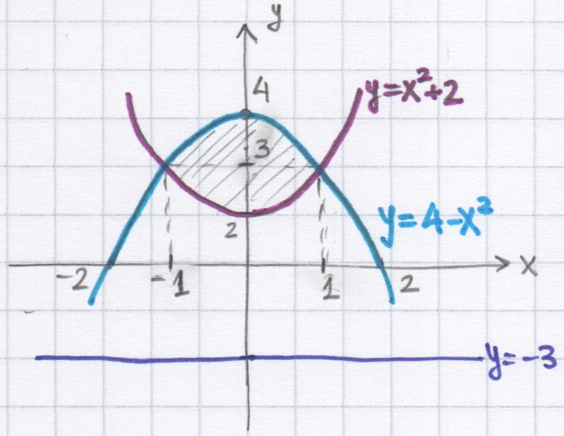
$R_{out} = 15 - y = 15 - (9 - x^2) = 6 + x^2$

$R_{in} = 3$

$$\text{Vol} = \pi \int_0^3 \left((6 + x^2)^2 - 9\right) dx = \pi \int_0^3 (36 + 12x^2 + x^4 - 9)\,dx = \pi \int_0^3 (27 + 12x^2 + x^4)\,dx$$

$$= \pi\left(27x + 4x^3 + \frac{x^5}{5}\right)\Big|_0^3 = \frac{1188}{5}\pi$$

③ Region b/w $f(x) = x^2+2$; $g(x) = 4-x^2$, about $y=-3$.



Intersection for curves?

$$x^2+2 = 4-x^2$$

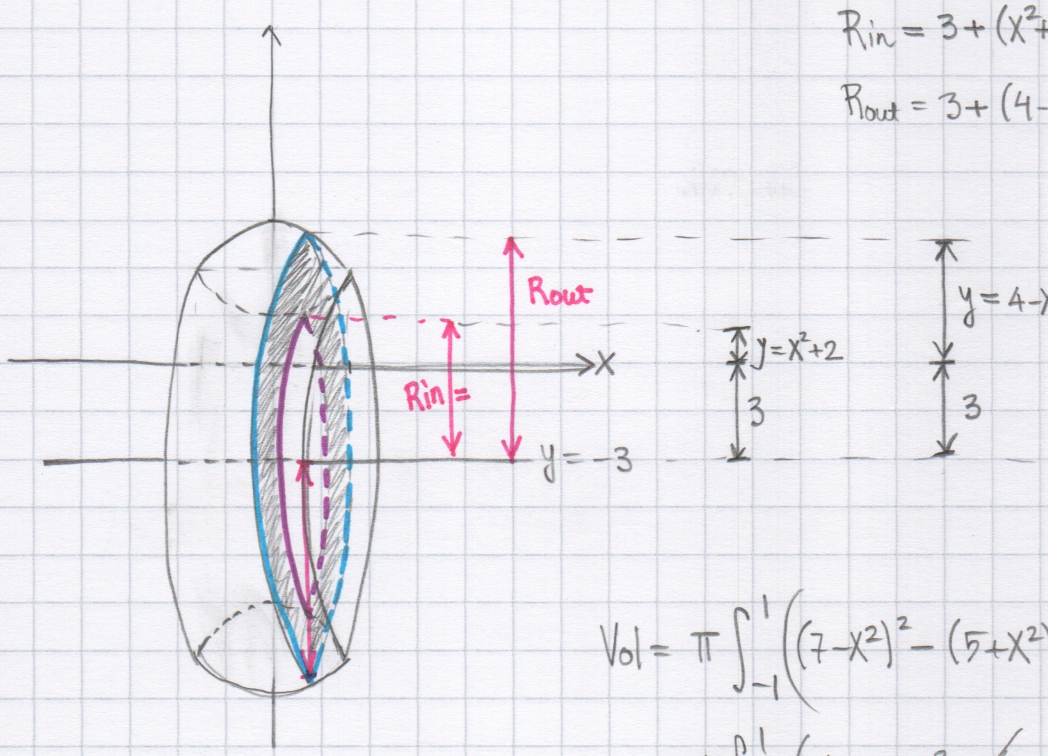
$$2x^2 = 2$$

$$x = \pm 1 \quad y=3$$

Cross-section = washer

$$R_{in} = 3 + (x^2+2) = 5+x^2$$

$$R_{out} = 3 + (4-x^2) = 7-x^2$$



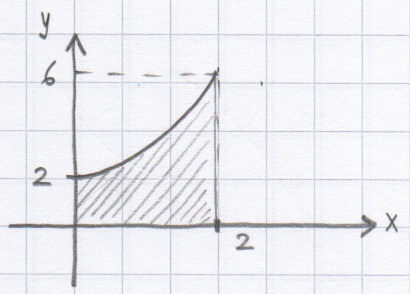
$$\text{Vol} = \pi \int_{-1}^{1} \left((7-x^2)^2 - (5+x^2)^2 \right) dx$$

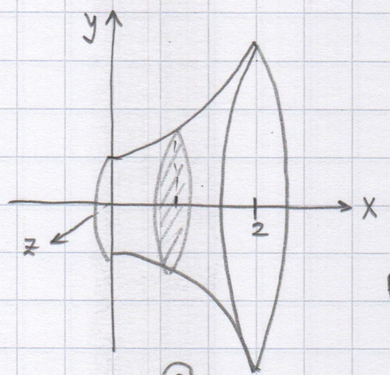
$$= \pi \int_{-1}^{1} \left(49 - 14x^2 + \cancel{x^4} - 25 - 10x^2 - \cancel{x^4} \right) dx$$

$$= \pi \int_{-1}^{1} \left(24 - 24x^2 \right) dx$$

$$= \pi \left(24x - 8x^3 \right) \Big|_{-1}^{1} = \pi \left(24 - 8 + 24 - 8 \right) = 32\pi$$

④ **Region under $y = x^2 + 2$, $0 \le x \le 2$.** ⓐ x-axis ; ⓑ $y = 6$





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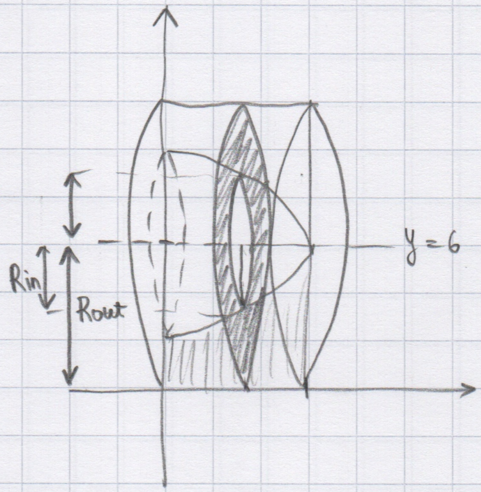


ⓐ $R(x) = y = x^2 + 2$

$$\text{Vol} = \int_0^2 \pi (x^2+2)^2\,dx = \pi \int_0^2 (x^4 + 4x^2 + 4)\,dx$$

$$= \pi \left(\frac{x^5}{5} + \frac{4x^3}{3} + 4x \right)\Big|_0^2 = \pi \left(\frac{32}{5} + \frac{32}{3} + 8 \right) = \pi \cdot \frac{376}{15}$$

$$32 \cdot \frac{8}{15} + 8 = 8 \cdot \frac{47}{15}$$

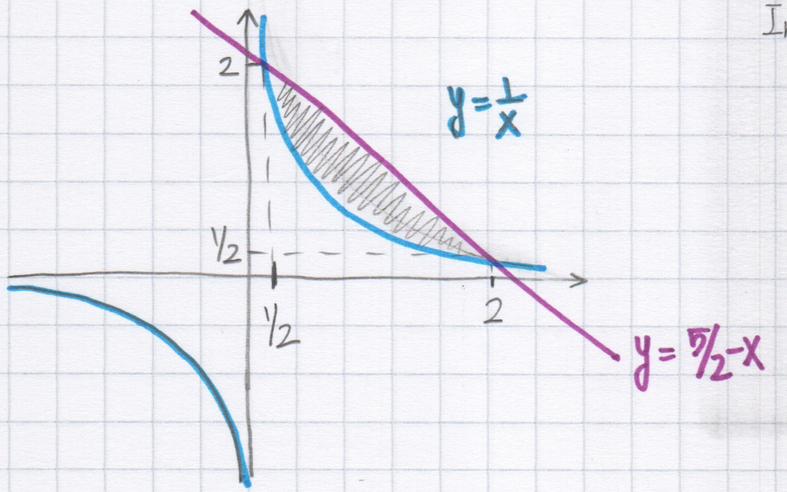
ⓑ $R_{in} = 6 - y = 6 - (x^2+2) = 4 - x^2$
$R_{out} = 6$

$$\text{Vol} = \int_0^2 \pi \cdot 36 - \pi (4-x^2)^2\,dx = \pi \int_0^2 (36 - 16 + 8x^2 - x^4)\,dx$$

$$= \pi \left(20x + \frac{8x^3}{3} - \frac{x^5}{5} \right)\Big|_0^2 = \pi \left(40 + \frac{64}{3} - \frac{32}{5} \right)$$

$$32 \cdot \frac{7}{15} \qquad = \pi \left(40 + \frac{224}{15} \right) = \pi \cdot \frac{824}{15}$$

⑤ $y=\frac{1}{x}$, $y=\frac{5}{2}-x$; y-axis.



Intersection?

$$\frac{1}{x}=\frac{5}{2}-x$$

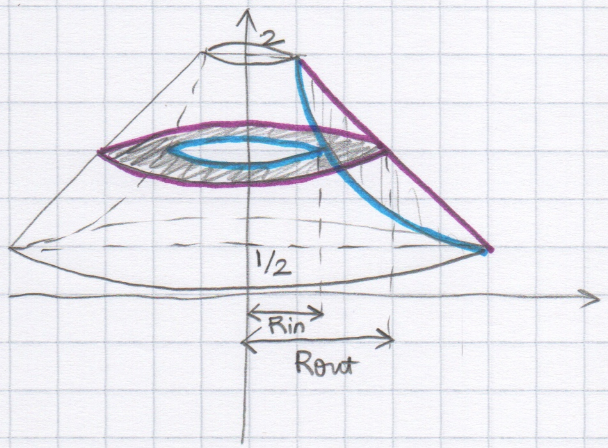
$$1=\frac{5}{2}x-x^2 \quad ; \quad x^2-\frac{5}{2}x+1=0$$

$$\Delta=\frac{25}{4}-4=\frac{9}{4}$$

$$x_{1,2}=\frac{5/2\pm 3/2}{2} \quad 2,\ 1/2$$

Pts of Intersection: $(1/2, \underline{\underline{2}})$

$(2, \underline{\underline{1/2}})$



Want to integrate dy

$$R_{out}=\frac{5}{2}-y$$

$$R_{in}=\frac{1}{y}$$

$$Vol=\pi\int_{1/2}^{2}\left(\left(\frac{5}{2}-y\right)^2-\left(\frac{1}{y}\right)^2\right)dy=\pi\int_{1/2}^{2}\left(\frac{25}{4}-5y+y^2-\frac{1}{y^2}\right)dy$$

$$=\pi\left(\frac{25}{4}y-\frac{5y^2}{2}+\frac{y^3}{3}+\frac{1}{y}\right)\Big|_{1/2}^{2}$$

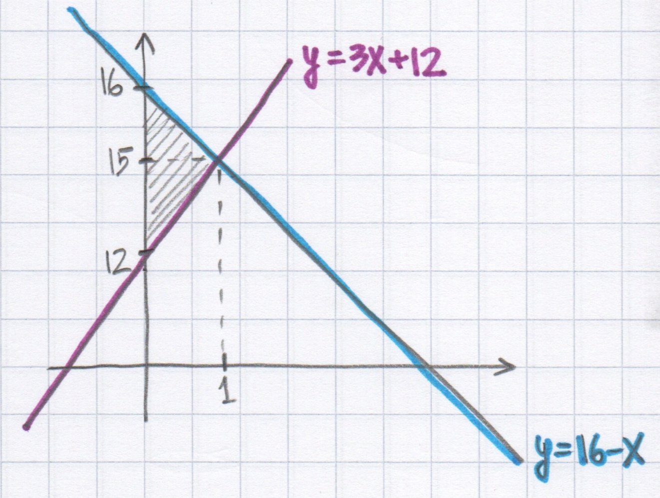
$$=\pi\left[\left(\frac{25}{2}-\frac{20}{2}+\frac{8}{3}+\frac{1}{2}\right)-\left(\frac{25}{8}-\frac{5}{8}+\frac{1}{24}+2\right)\right]$$

$$=\pi\left(3+\frac{8}{3}-\frac{61}{24}-2\right)=\pi\left(1+\frac{3}{24}\right)=\pi\left(1+\frac{1}{8}\right)=\frac{9\pi}{8}$$

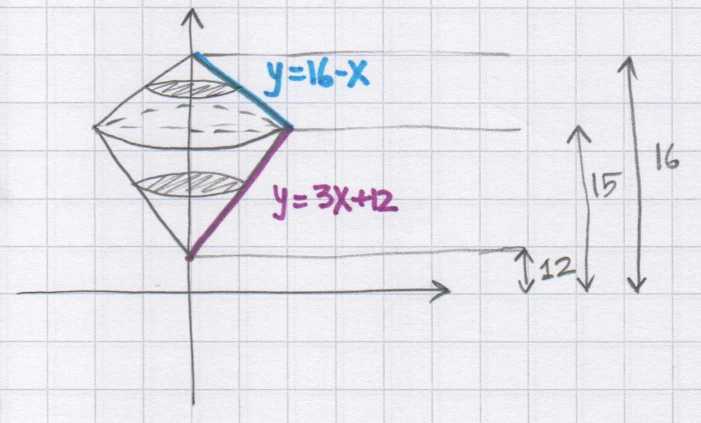
6) $y=16-x,\ y=3x+12,\ x=0\ ;$ y-axis.

$16-x=3x+12$

$4=4x \quad x=1 \quad$ Pt. of intersection: $(1,15)$.



Have to split volume in two!



(Top part): $\mathrm{Vol}_1 = \int_{15}^{16} \pi(16-y)^2\,dy = \pi\int_{15}^{16}(16^2-32y+y^2)\,dy$

$= \pi\left(16^2y-16y^2+\frac{y^3}{3}\right)\Big|_{15}^{16} = \pi\left(16^2\cdot16-16^3+\frac{16^3}{3}-16^2\cdot15+16\cdot15^2-\frac{15^3}{3}\right)$

$= \pi\left(16^2-16(16^2-15^2)+\frac{1}{3}(16^2+16\cdot15+15^2)\right)$

$= \pi\left(16^2-16\cdot31+\frac{1}{3}(16^2+16\cdot15+15^2)\right) = \frac{\pi}{3}(15^2-16\cdot14) = \frac{\pi}{3}$

(Bottom part): $\mathrm{Vol}_2 = \int_{12}^{15}\pi\left(\frac{1}{3}(y-12)\right)^2dy = \frac{\pi}{9}\int_{12}^{15}(y^2-24y+12^2)\,dy$

$= \frac{\pi}{9}\left(\frac{y^3}{3}-12y^2+12^2y\right)\Big|_{12}^{15}$

$= \frac{\pi}{9}\left(\frac{15^3}{3}-12\cdot15^2+12^2\cdot15-\frac{12^3}{3}+\cancel{12^3}-\cancel{12^3}\right)$

$= \frac{\pi}{9}\left(15^2+15\cdot12+12^2-15\cdot12\cdot3\right) = \frac{\pi}{9}(15-12)^2 = \pi$

$\Rightarrow$ Total Vol $= \frac{4\pi}{3}$