11.9 Representing Functions as Power Series

I. We have seen before that:

$$\frac{1}{1-x} = 1+x+x^2+x^3+\ldots = \sum_{n=0}^{\infty} x^n \text{ for all } |x|<1$$

1). Find a power series representation for the function: $f(x)=\frac{1}{7+x}$ and determine the interval of convergence.

$$\frac{1}{7+x} = \frac{1}{7}\cdot\frac{1}{1+\frac{x}{7}} = \frac{1}{7}\cdot\frac{1}{1-\left(-\frac{x}{7}\right)}$$

$$= \frac{1}{7}\cdot\sum_{n=0}^{\infty}\left(-\frac{x}{7}\right)^n \quad \text{for all } \left|-\frac{x}{7}\right|<1$$

$$= \frac{1}{7}\cdot\sum_{n=0}^{\infty}\frac{(-1)^n\cdot x^n}{7^n} = \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n\cdot x^n}{7^{n+1}}} \quad \text{for all } |x|<7$$

Interval of convergence: $\left|-\frac{x}{7}\right|<1$; $|x|<7$; $\boxed{(-7,7)}$

2). Same for $f(x)=\frac{x}{64+x^2}$

$$\frac{x}{64+x^2} = \frac{x}{64}\cdot\frac{1}{1+\frac{x^2}{64}} = \frac{x}{64}\cdot\frac{1}{1-\left(-\frac{x^2}{64}\right)}$$

$$= \frac{x}{64}\cdot\sum_{n=0}^{\infty}\left(-\frac{x^2}{64}\right)^n \quad \text{for all } \left|\frac{-x^2}{64}\right|<1$$

$$= \frac{x}{64}\cdot\sum_{n=0}^{\infty}\frac{(-1)^n\cdot x^{2n}}{8^{2n}} \quad \text{for all } |x^2|<64$$

$$= \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n\cdot x^{2n+1}}{64^{n+1}}} \quad \text{for all } x \text{ in } \boxed{(-8,8)}$$

3). Same for $f(x)=\frac{12+x}{1-x}$

$$\frac{12+x}{1-x} = (12+x)\cdot\frac{1}{1-x} = (12+x)\cdot\sum_{n=0}^{\infty}x^n \quad \text{for all } |x|<1$$

$$= \boxed{\sum_{n=0}^{\infty}(12+x)\cdot x^n} \quad \text{for all } x \text{ in } \boxed{(-1,1)}$$

$\boxed{\text{II}}$. Differentiation & Integration of Power Series:

Theorem:

If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence $R>0$, then the function:

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

is differentiable on $(a-R, a+R)$ and:

1). $\boxed{f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}} = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots$

2). $\boxed{\int f(x)\,dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}} = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \dots$

The radii of convergence of both series in 1) and 2) are both $\underline{\underline{R}}$.

1). Find a power series representation for $f(x) = \ln(9-x)$ and find the radius of convergence.

$$f(x) = \ln(9-x) = \int -\frac{1}{9-x}\,dx = -\int \frac{1}{9-x}\,dx = -\frac{1}{9}\int \frac{1}{1-\frac{x}{9}}\,dx$$

$$= -\frac{1}{9}\int \sum_{n=0}^{\infty}\left(\frac{x}{9}\right)^n dx \quad \text{for } \left|\frac{x}{9}\right|<1, \text{ so } |x|<9 \qquad \boxed{R=9}$$

$$= -\frac{1}{9}\sum_{n=0}^{\infty}\int \frac{x^n}{9^n}\,dx = -\frac{1}{9}\sum_{n=0}^{\infty}\frac{1}{9^n}\int x^n\,dx$$

$$= \left(-\frac{1}{9}\sum_{n=0}^{\infty}\frac{1}{9^n}\cdot\frac{x^{n+1}}{n+1}\right) + C = C - \sum_{n=0}^{\infty}\frac{1}{9^{n+1}}\cdot\frac{x^{n+1}}{n+1}$$

$\Rightarrow \quad f(x) = C - \sum_{n=1}^{\infty}\frac{x^n}{9^n \cdot n}$ for all x in $(-9,9)$

To find C, let $x=0$ for example. Then $f(0) = \ln(9)$, so $C = \ln(9)$. So

$$\boxed{f(x) = \ln(9) - \sum_{n=1}^{\infty}\frac{x^n}{9^n \cdot n}}$$

2). Same for $f(x) = \frac{x}{(1+9x)^2}$ (WebAssign 11.9 problem 6)

We know that: $\frac{1}{1+9x} = \frac{1}{1-(-9x)} = \sum_{n=0}^{\infty} (-9x)^n$ for all $|-9x|<1$, so $|x|<\frac{1}{9}$

$\boxed{R=\frac{1}{9}}$

So:

$$\left(\frac{1}{1+9x}\right)' = \sum_{n=0}^{\infty} \left((-9)^n \cdot x^n\right)'$$

$$-\frac{9}{(1+9x)^2} = \sum_{n=1}^{\infty} (-9)^n \cdot n x^{n-1} = \sum_{n=0}^{\infty} (-9)^{n+1} \cdot (n+1) \cdot x^n$$

$$\Rightarrow \quad \frac{1}{(1+9x)^2} = \sum_{n=0}^{\infty} (-9)^n (n+1) \cdot x^n$$

$$\Rightarrow \quad \frac{x}{(1+9x)^2} = \boxed{\sum_{n=0}^{\infty} (-9)^n (n+1) \cdot x^{n+1}} \quad \text{for all } x \text{ in } \left(-\frac{1}{9}, \frac{1}{9}\right).$$

3). Evaluate $f(t) = \int \frac{t}{1-t^7}\,dt$ as a power series; find the radius of converge

$$\frac{t}{1-t^7} = t \cdot \frac{1}{1-t^7} = t \cdot \sum_{n=0}^{\infty} (t^7)^n = t \cdot \sum_{n=0}^{\infty} t^{7n} = \sum_{n=0}^{\infty} t^{7n+1} \quad \text{for all } |t|<1.$$

$\boxed{R=1}$

$$\Rightarrow \quad \int \frac{t}{1-t^7}\,dt = \boxed{C + \sum_{n=0}^{\infty} \frac{t^{7n+2}}{7n+2}}$$

4). Let: $f(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^2}$

a). Find the interval of convergence for f.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)^2}\cdot\frac{n^2}{x^n}\right| = |x|$$

By the Ratio Test, the series converges on $(-1,1)$ but we must test the endpoints:

- $x=-1 \Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ converges by the Alternating Series Test
- $x=1 \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2}$ convergent p-series $(p=2>1)$

So $\boxed{I=[-1,1]}$

b). Find the interval of convergence for f'.

We know that f', f'' have the same radius of convergence as f, in this case 1. So we must test the endpoints 1, −1.

$$f'(x)=\sum_{n=1}^{\infty}\frac{nx^{n-1}}{n^2}=\sum_{n=1}^{\infty}\frac{x^{n-1}}{n}=\sum_{n=0}^{\infty}\frac{x^n}{n+1}$$

- $x=-1 \Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ converges by the Alternating Series Test.
- $x=1 \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n}$ diverges (harmonic series).

So $\boxed{I=[-1,1)}$

c). Same for f''

$$f''(x)=\left(\sum_{n=0}^{\infty}\frac{x^n}{n+1}\right)'=\sum_{n=1}^{\infty}\frac{nx^{n-1}}{n+1}$$

- $x=-1 \Rightarrow \sum_{n=1}^{\infty}\frac{n(-1)^{n-1}}{n+1}$ diverges by the Test for Divergence: $\lim_{n\to\infty}\frac{(-1)^{n-1}n}{n+1}$ DNE
- $x=1 \Rightarrow \sum_{n=1}^{\infty}\frac{n}{n+1}$ diverges by the Test for Divergence: $\lim_{n\to\infty}\frac{n}{n+1}=1\neq 0.$

So $\boxed{I=(-1,1)}$

5).

a). Starting with the geometric series $\sum_{n=0}^{\infty} x^n$, find the sum of the series:

$$\sum_{n=1}^{\infty} n x^{n-1} \; ; \; |x|<1$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \text{ for all } |x|<1 \Rightarrow \left(\frac{1}{1-x}\right)' = \sum_{n=1}^{\infty} n x^{n-1} \text{ for all } |x|<1$$

$$\Rightarrow \sum_{n=1}^{\infty} n x^{n-1} = \boxed{\frac{1}{(1-x)^2}} \; , |x|<1.$$

b). Use the result above to find:

(i). $\sum_{n=1}^{\infty} n x^n \; ; \; |x|<1$

$$\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2} \Rightarrow \sum_{n=1}^{\infty} n x^n = \boxed{\frac{x}{(1-x)^2}}$$

(ii). $\sum_{n=1}^{\infty} \frac{n}{7^n}$

$$\sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2} \Rightarrow \text{for } x = \tfrac{1}{7} : \sum_{n=1}^{\infty} n \cdot \left(\tfrac{1}{7}\right)^n = \frac{1/7}{(1-1/7)^2} = \boxed{\frac{7}{36}}$$

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c). Find the sum of the series:

(i). $\sum_{n=2}^{\infty} n(n-1) x^n \; ; \; |x|<1$

$$\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2} \Rightarrow \sum_{n=2}^{\infty} n(n-1) x^{n-2} = \left(\frac{1}{(1-x)^2}\right)' = +\frac{2}{(1-x)^3}$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) x^n = \boxed{\frac{2x^2}{(1-x)^3}}$$

(ii). $\sum_{n=2}^{\infty} \frac{n^2-n}{4^n} = \sum_{n=2}^{\infty} \frac{n(n-1)}{4^n}$; let $x = \frac{1}{4}$ above

$$= \frac{2 \cdot (1/4)^2}{(1-1/4)^3} = \boxed{\frac{8}{27}}$$

(iii). $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

(b)(i) with $x = \frac{1}{2}$: $\sum_{n=1}^{\infty} n \cdot \left(\frac{1}{2}\right)^n = \frac{1/2}{(1-1/2)^2} = 2$

(c)(i) with $x = \frac{1}{2}$: $\sum_{n=2}^{\infty} (n^2-n)\left(\frac{1}{2}\right)^n = \frac{2(1/2)^2}{(1-1/2)^3} = 4$

$$= \frac{1}{2} + \sum_{n=2}^{\infty} \frac{n^2}{2^n} = \frac{1}{2} + \left(2 - \frac{1}{2}\right) + 4 = 2+4 = \boxed{6}$$

6). a). Find a power series representation for $f(x) = \ln(1+x)$

$$f(x) = \ln(1+x) = \int \frac{1}{1+x}\,dx = \int \frac{1}{1-(-x)}\,dx = \int \sum_{n=0}^{\infty} (-x)^n\,dx \quad , |x|<1$$

$$= \left(\sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{n+1}}{n+1} \right) + C$$

$$f(0) = 0 \Rightarrow C = 0 \Rightarrow \quad f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = \boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}}$$

b). Use part a). to find a power series for $f(x) = x \ln(1+x)$

$$x \ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot x^{n+1}}{n} \quad , |x|<1$$

$$= \sum_{n=2}^{\infty} \frac{(-1)^{n-2} \cdot x^n}{n-1} = \sum_{n=2}^{\infty} \frac{(-1)^n \cdot x^n}{n-1}$$

c). Same for $f(x) = \ln(x^2+1)$:

$$\ln(x^2+1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot (x^2)^n}{n} \quad , |x^2|<1$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n}}{n} \quad , |x|<1.$$