11.8 Power Series

Def: A power series is a series of the form:

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

where x is a variable and the c_n's are constants (called the "coefficients" of the series).

- For each fixed x, the series above is a series of constants we may test for convergence or divergence.
- The series above may converge for some values of x and diverge for others.
- The function:

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

has as its domain the values of x for which the series converges, called the interval of convergence.

Def.: A power series centered at a or ("about a") where a is a constant is a power series of the form:

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \dots$$

R = radius of convergence

I = interval of convergence

Theorem:

For a given power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ there are 3 possibilities:

1). The series converges only when $x=a$. $[R=0;\ I=\{a\}]$

2). The series converges for all x $[R=\infty;\ I=(-\infty,\infty)]$

3). There is a number $R>0$ such that the series converges for $|x-a|<R$ and diverges for $|x-a|>R$. $[R;\ I=(a-R, a+R)$ or $(a-R, a+R]$ or $[a-R, a+R)$ or $[a-R, a+R]]$.

Remark: In case 3)., anything can happen at the endpoints $a-R$ and $a+R$ (must be tested individually!)

Example 1: $\sum_{n=1}^{\infty} 4(-1)^n \cdot n x^n$

$a_n = 4(-1)^n n x^n$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{4(n+1)|x|^{n+1}}{4n|x|^n} = \frac{n+1}{n}|x| \xrightarrow[n\to\infty]{} |x|$$

$\Rightarrow \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x| < 1$: By the Ratio Test, the series converges absolutely for $|x|<1$ and diverges for $|x|>1$.

Endpoints? $x=-1$: $\sum_{n=1}^{\infty} 4(-1)^n \cdot n(-1)^n = \sum_{n=1}^{\infty} 4n$ diverges by Divergence Test $\left(\lim_{n\to\infty}(4n) = \infty\right)$

$x=1$: $\sum_{n=1}^{\infty} 4(-1)^n \cdot n \cdot 1^n = \sum_{n=1}^{\infty} 4(-1)^n \cdot n$ diverges by Div. Test $\left(\lim_{n\to\infty}((-1)^n \cdot n) \text{ DNE}\right)$

$\Rightarrow$ Radius of convergence $\boxed{R=1}$ Interval of convergence $\boxed{I=(-1,1)}$

Example 2: $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$

$a_n = \frac{(x-3)^n}{n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(\frac{|x-3|^{n+1}}{n+1}\cdot\frac{n}{|x-3|^n}\right) = \lim_{n\to\infty}\left(\frac{n}{n+1}|x-3|\right) = |x-3| < 1$$

$\Rightarrow$ By the Ratio Test, the series is absolutely convergent for $|x-3|<1$ and divergent for $|x-3|>1$. Radius of convergence $\boxed{R=1}$

$|x-3|<1 \iff -1 < x-3 < 1 \iff \boxed{2<x<4}$

Endpoints? $x=2$: $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ convergent! (Alternating harmonic series)

$x=4$: $\sum_{n=1}^{\infty}\frac{1}{n}$ divergent! (Harmonic series)

$\Rightarrow$ Interval of convergence $\boxed{[2,4)}$

Example 3: $\sum_{n=0}^{\infty} n!\, x^n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)!\,|x|^{n+1}}{n!\,|x|^n} = \lim_{n\to\infty}(n+1)|x| = \begin{cases}\infty & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}$$

$\boxed{R=0,\ I=\{0\}}$

$\Rightarrow$ By the Ratio Test, the series only converges when $x=0$, and diverges otherwise.

Example 4: $\sum_{n=1}^{\infty}\frac{(x-4)^n}{n^n}$ $\quad a_n = \left(\frac{x-4}{n}\right)^n \Rightarrow$ suited for Root Test!

$$\sqrt[n]{|a_n|} = \frac{|x-4|}{n} \xrightarrow[n\to\infty]{} 0 \text{ for all } x \Rightarrow \lim_{n\to\infty}\sqrt[n]{|a_n|} = 0 < 1 \text{ for all } x$$

$\Rightarrow$ By the Root Test, the series converges for all real x.

$\boxed{R=\infty,\ I=(-\infty,\infty)=\mathbb{R}}$