Alternating Series

= Series whose terms are alternatively positive and negative, such as:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$$

Alternating Series Test : (AST)

If the series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - \dots$ where every $\boxed{b_n > 0}$ satisfies:

(1). $b_{n+1} \leq b_n$ for all n (decreasing sequence)

(2). $\lim_{n \to \infty} b_n = 0$

then the series converges.

Example: The Alternating Harmonic Series (converges):

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$

The sequence $b_n = \frac{1}{n}$ is decreasing:

$$b_{n+1} = \frac{1}{n+1} \leq \frac{1}{n} = b_n$$

and $\lim_{n \to \infty} \frac{1}{n} = 0$, so the series converges by AST.

Remarks:

1). This test also applies to series like $\sum_{n=1}^{\infty} (-1)^n b_n$, $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ etc.

2). This test cannot be used to show that a series diverges!

Exercises:

1). $\frac{5}{6} - \frac{5}{8} + \frac{5}{10} - \frac{5}{12} + \frac{5}{14} - \dots = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{5}{2(n+2)}$

- $b_n = \frac{5}{2(n+2)} \geqslant \frac{5}{2(n+3)} = b_{n+1}$ so $\{b_n\}$ is decreasing
- $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{5}{2(n+2)} = 0$

So the series converges by AST.

2). $\sum_{n=1}^{\infty} 5(-1)^n e^{-n}$

- $b_n = 5e^{-n} = \frac{5}{e^n} \geqslant \frac{5}{e^{n+1}} = b_{n+1}$ so $\{b_n\}$ is decreasing
- $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{5}{e^n} = 0$

So the series converges by AST.

3). $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{n}{13^n}$

$b_n = \frac{n}{13^n} \boxed{\geqslant} \frac{n+1}{13^{n+1}} = b_{n+1}$

$13n \boxed{\geqslant} n+1$

$12n \boxed{\geqslant} 1$

So $\{b_n\}$ is decreasing, and $\lim_{n\to\infty} \frac{n}{13^n} = 0$

So the series converges by AST.

4). $\sum_{n=1}^{\infty} (-1)^{n-1} \cdot e^{3/n}$

$b_n = e^{3/n} \Rightarrow \lim_{n\to\infty} e^{3/n} = e^0 = 1$, so AST does not apply.

However, since $\lim_{n\to\infty} e^{3/n} = 1 \neq 0$, we have that:

$$\lim_{n\to\infty} (-1)^{n-1} e^{3/n} \text{ DNE}$$

so the series diverges by the Test for Divergence.

$$\boxed{\lim_{n\to\infty} b_n = c \neq 0} \Rightarrow \boxed{\begin{array}{l} \lim_{n\to\infty} (-1)^n b_n \text{ DNE} \\ \lim_{n\to\infty} (-1)^{n-1} b_n \text{ DNE} \\ \text{etc.} \end{array}}$$

5). $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{3+2\sqrt{n}}$

$b_n = \frac{\sqrt{n}}{3+2\sqrt{n}}$ and $\lim_{n\to\infty} \frac{\sqrt{n}}{3+2\sqrt{n}} = \frac{1}{2} \neq 0$

so AST does not apply, but

$$\lim_{n\to\infty} (-1)^n \cdot \frac{\sqrt{n}}{3+2\sqrt{n}} \text{ DNE}$$

so the series diverges by the Test for Divergence.