The Integral Test :

Suppose f is a continuous, positive, decreasing function on $[1,\infty)$ and consider the sequence $a_n = f(n)$. Then:

$$\boxed{\int_1^\infty f(x)\,dx \text{ converges}} \Rightarrow \boxed{\sum_{n=1}^\infty a_n \text{ converges}}$$

$$\boxed{\int_1^\infty f(x)\,dx \text{ diverges}} \Rightarrow \boxed{\sum_{n=1}^\infty a_n \text{ diverges}}$$

- As a consequence, the p-series : $\boxed{\sum_{n=1}^\infty \frac{1}{n^p}}$ converges for $p>1$, and diverges for $p\leq 1$.

- CAUTION : The Integral Test does not tell us what the series converges to, i.e. if:

$$\int_1^\infty f(x)\,dx = \lim_{t\to\infty}\int_1^t f(x)\,dx = a < \infty$$

we know that $\sum_{n=1}^\infty f(n)$ converges, but it does not necessarily converge to a!

Exercises :

1). $\sum_{n=1}^\infty \frac{9}{\sqrt[5]{n}} = 9\sum_{n=1}^\infty \frac{1}{\sqrt[5]{n}} = 9\cdot\sum_{n=1}^\infty \frac{1}{n^{1/5}}$ divergent p-series $(p=\frac{1}{5}<1)$

2). $1+\frac{1}{16}+\frac{1}{81}+\frac{1}{256}+\frac{1}{625}+\dots = 1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\dots$

$= \sum_{n=1}^\infty \frac{1}{n^4}$ convergent p-series $(p=4>1)$

3). $\sum_{n=1}^\infty \frac{\sqrt{n}+4}{n^2} = \sum_{n=1}^\infty\left(\frac{\sqrt{n}}{n^2}+\frac{4}{n^2}\right) = \sum_{n=1}^\infty\left(\frac{1}{n^{3/2}}+\frac{4}{n^2}\right)$ convergent

as the sum of two convergent p-series (one with $p=\frac{3}{2}>1$ and the other with $p=2>1$).

4). $\sum_{n=1}^{\infty} \frac{n}{n^2+2}$

$f(x) = \frac{x}{x^2+2}$ is continuous, decreasing & positive on $[1,\infty)$.

$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{x}{x^2+2}\,dx = \frac{1}{2}\ln(x^2+2)\Big|_1^{\infty} = \infty$

So the series diverges by the Integral Test.

5). $\frac{1}{3} + \frac{1}{7} + \frac{1}{11} + \frac{1}{15} + \frac{1}{19} + \dots = \sum_{n=1}^{\infty} \frac{1}{4n-1}$

$f(x) = \frac{1}{4x-1}$ is continuous, decreasing & positive on $[1,\infty)$

$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{1}{4x-1}\,dx = \frac{1}{4}\ln(4x-1)\Big|_1^{\infty} = \infty$

So the series diverges by the Integral Test.

6). $\sum_{n=1}^{\infty} ne^{-2n}$

$f(x) = xe^{-2x}$ is continuous, decreasing & positive on $[1,\infty)$.

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} xe^{-2x}\,dx = \int_1^{\infty} x\left(\frac{e^{-2x}}{-2}\right)'dx$$

$$= x\,\frac{e^{-2x}}{-2}\Big|_1^{\infty} - \int_1^{\infty} \frac{e^{-2x}}{-2}\,dx$$

$$= 0 - \frac{e^{-2}}{-2} - \frac{e^{-2x}}{4}\Big|_1^{\infty} = \frac{e^{-2}}{2} - \left(0 - \frac{e^{-2}}{4}\right)$$

$$= \frac{e^{-2}}{2} + \frac{e^{-2}}{4} = \frac{3e^{-2}}{4} < \infty$$

So the series converges by the Integral Test.

Caution: The series does not converge to $\frac{3e^{-2}}{4}$, the Integral Test just tells us that it converges.

7). What are the conclusions of the Integral Test for $\sum_{n=1}^{\infty} \frac{9\cos(\pi n)}{\sqrt{n}}$?

None! The function $f(x) = \frac{9\cos(\pi x)}{\sqrt{x}}$ is not positive on $[1,\infty)$, so I.T. does not apply!