Test for Divergence

Thm.: If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Proof: If $\sum_{n=1}^{\infty} a_n$ converges, then $\{S_N\}$ converges to some number S.
Then:
$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} S_{N-1} = S$$
$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty}(S_n - S_{n-1}) = S - S = 0.$$
∎

As a consequence:

The Test for Divergence: $\boxed{\text{If } \lim_{n\to\infty} a_n \neq 0, \text{ then } \sum_{n=1}^{\infty} a_n \text{ diverges.}}$

! CAUTION ! This test can only be used to conclude that a series diverges.

If $\lim_{n\to\infty} a_n = 0$, then it's possible for $\sum_{n=1}^{\infty} a_n$ to converge or diverge:

Example 1: $\sum_{n=1}^{\infty} \frac{1}{2^n}$ convergent geometric series.
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2^n} = 0$$

Example 2: $\sum_{n=1}^{\infty} \frac{1}{n}$ divergent (harmonic series)
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Exercises:

1). $\sum_{n=1}^{\infty} \frac{n-1}{4n+1}$ diverges by the Test for Divergence, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n-1}{4n+1} = \frac{1}{4}$

2). $\sum_{n=1}^{\infty} \arctan(2n)$ diverges by the Test for Divergence, since $\lim_{n\to\infty} \arctan(2n) = \frac{\pi}{2}$

3). $\sum_{n=1}^{\infty} \ln\left(\frac{n^2+6}{7n^2+2}\right)$ diverges by the Test for Divergence, since $\lim_{n\to\infty} \ln\left(\frac{n^2+6}{7n^2+2}\right) = \ln\frac{1}{7}$

4). $\sum_{n=1}^{\infty} \frac{1}{n^2}$? $\lim_{n\to\infty} \frac{1}{n^2} = 0$, so the Test for Divergence is inconclusive.

Telescoping Series

1). $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$ $\quad \lim_{n\to\infty} a_n = \ln(1) = 0 \Rightarrow$ possible to converge or diverge

$$= \sum_{n=1}^{\infty} \ln(n) - \ln(n+1)$$

$$\begin{aligned} S_N = {} & \ln(1) - \cancel{\ln(2)} \\ & + \cancel{\ln(2)} - \cancel{\ln(3)} \\ & + \cancel{\ln(3)} - \cancel{\ln(4)} \\ & + \\ & \vdots \\ & + \cancel{\ln(N)} - \ln(N+1) = \ln(1) - \ln(N+1) = -\ln(N+1) \xrightarrow[N\to\infty]{} -\infty \end{aligned}$$

So $\lim_{N\to\infty} S_N = -\infty$, so the series <u>diverges.</u>

2). $\sum_{n=1}^{\infty} \left(\cos\frac{7}{n^2} - \cos\frac{7}{(n+1)^2}\right)$ $\quad \lim_{n\to\infty} a_n = \cos(0) - \cos(0) = 0$

Test for Divergence Inconclusive

$$\begin{aligned} S_N = \sum_{n=1}^{N} \left(\cos\frac{7}{n^2} - \cos\frac{7}{(n+1)^2}\right) = {} & \cos\frac{7}{1} - \cancel{\cos\frac{7}{2^2}} \\ & + \cancel{\cos\frac{7}{2^2}} - \cancel{\cos\frac{7}{3^2}} \\ & + \cancel{\cos\frac{7}{3^2}} - \cancel{\cos\frac{7}{4^2}} \\ & + \quad\quad - \\ & \vdots \\ & + \cancel{\cos\frac{7}{N^2}} - \cos\frac{7}{(N+1)^2} \end{aligned}$$

$$= \cos(7) - \cos\frac{7}{(N+1)^2}$$

$$\lim_{N\to\infty} S_N = \cos(7) - \cos(0) = \cos(7) - 1 \quad \Rightarrow \quad \boxed{\sum_{n=1}^{\infty} a_n = \cos(7) - 1}$$

(convergent)