Thm: If f has a power series representation (expansion) at a, i.e. if:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \quad |x-a| < R$$

Then the coefficients are given by:

$$\boxed{c_n = \frac{f^{(n)}(a)}{n!}}$$

- $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots \qquad |x-a|<R$

  $x=a \Rightarrow f(a) = c_0 = \dfrac{f^{(0)}(a)}{0!}$

- $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \dots$

  $x=a \Rightarrow f'(a) = c_1 = \dfrac{f^{(1)}(a)}{1!}$

- $f''(x) = 2c_2 + 3\cdot 2c_3(x-a) + 4\cdot 3c_4(x-a)^2 + \dots$

  $x=a \Rightarrow f''(a) = 2c_2 \Rightarrow c_2 = \dfrac{f''(a)}{2} = \dfrac{f^{(2)}(a)}{2!}$

- $f^{(3)}(x) = 3\cdot 2c_3 + 4\cdot 3\cdot 2c_4(x-a) + 5\cdot 4\cdot 3c_5(x-a)^2 + \dots$

  $x=a \Rightarrow f^{(3)}(a) = 3\cdot 2c_3 \Rightarrow c_3 = \dfrac{f^{(3)}(a)}{3!}$

$\Rightarrow$

$$f(x) = \boxed{\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n}$$ The Taylor series of f at a (centered at a)

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

- For the special case $x=0$:

$$f(x) = \boxed{\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n}$$ The Maclaurin series of f

$$= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \dots$$

1). If $f^{(n)}(0) = (n+1)!$ for $n = 0, 1, 2, \ldots$ find the Maclaurin series of f:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{(n+1)!}{n!} x^n = \boxed{\sum_{n=0}^{\infty} (n+1) x^n}$$

Radius of convergence?

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+2)\, x^{n+1}}{(n+1)\, x^n} \right| = |x| \qquad \boxed{R = 1}$$

2). Find the Taylor series for f centered at 2 if: $f^{(n)}(2) = \dfrac{(-1)^n \cdot n!}{8^n (n+3)}$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!} (x-2)^n = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{8^n (n+3)} (x-2)^n}$$

Radius of convergence?

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} \cdot (x-2)^{n+1}}{8^{n+1} (n+4)} \cdot \frac{8^n (n+3)}{(-1)^n (x-2)^n} \right| = \frac{|x-2|}{8} < 1$$

$$|x-2| < 8 \Rightarrow \boxed{R = 8}$$

- If a function f has a power series representation at a, then f is equal to the sum of its Taylor series.
- But there are functions that are not equal to their Taylor series.
- Under what circumstances is a function equal to the sum of its Taylor series? If f has derivatives of all orders, when is it true that:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \quad ?$$

- By definition of convergent series, $f(x)$ must be the limit of the partial sums of the series:

The n^{th} degree Taylor polynomial of f at a

$$\boxed{T_n(x) = \sum_{i=1}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i}$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-$$

- So $f(x)$ is the sum of its Taylor Series if:

$$f(x) = \lim_{n\to\infty} T_n$$

- For every n, let

$$\boxed{R_n(x) = f(x) - T_n(x)}$$ The remainder of the Taylor series

- Note that if $\lim_{n\to\infty} R_n = 0$, then $\lim_{n\to\infty}[f(x) - T_n(x)] = \lim_{n\to\infty} R_n = 0$, and then $f(x) = \lim_{n\to\infty} T_n(x)$, so then the function $f(x)$ equals the sum of its Taylor Series.

Thm: If $T_n(x)$ is the n^{th} degree Taylor polynomial of f at a, and $R_n(x) = f(x) - T_n(x)$ is the n^{th} remainder of the Taylor series, and:

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x-a| < R$, then f equals the sum of its Taylor series on $|x-a| < R$.

1). $\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$ for all real x

$f(x) = e^x \Rightarrow f^{(n)}(x) = e^x \Rightarrow f^{(n)}(0) = 1$ for all n

$$\Rightarrow \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1$ for all real x.

Taylor's Inequality:

If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor Series satisfies:

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1} \text{ for } |x-a| \leq d$$

- This is the result one usually uses to prove that $\lim_{n\to\infty} R_n = 0$. For example:
  - If $f(x) = e^x$, then $f^{(n+1)}(x) = e^x$ for all n.
  - If d is any positive number and $|x| \leq d$, then $|f^{(n+1)}(x)| = |e^x| \leq e^d = M$

$$\lim_{n\to\infty} \frac{e^d}{(n+1)!} |x|^{n+1} = e^d \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

By the Squeeze Theorem, $\lim_{n\to\infty} |R_n(x)| = 0$, and then $\lim_{n\to\infty} R_n(x) = 0$.

2). $\boxed{\sin(x) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!}}$ for all real x.

$$f(x) = \sin(x)\ ;\ f^{(0)}(0) = 0$$
$$f'(x) = \cos(x)\ ;\ f^{(1)}(0) = 1$$
$$f''(x) = -\sin(x)\ ;\ f^{(2)}(0) = 0$$
$$f^{(3)}(x) = -\cos(x)\ ;\ f^{(3)}(0) = -1$$
$$f^{(4)}(x) = \sin(x)\ ;\ f^{(4)}(0) = 0$$

Maclaurin series: $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots$

3). $\boxed{\cos(x) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!}}$ for all real x.

$$\cos(x) = (\sin(x))' = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{(2n+1)\, x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!}$$

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4). Find the Maclaurin series of $f(x) = \ln(1+5x)$

$$f(x) = \ln(1+5x) \Rightarrow f(0) = 0$$
$$f'(x) = \frac{5}{1+5x} \Rightarrow f'(0) = 5$$
$$f''(x) = -\frac{5^2}{(1+5x)^2} \Rightarrow f''(0) = -5^2$$
$$f'''(x) = +\frac{2 \cdot 5^3}{(1+5x)^3} \Rightarrow f^{(3)}(0) = 2 \cdot 5^3$$
$$f^{(4)}(x) = -\frac{2 \cdot 3 \cdot 5^4}{(1+5x)^4} \Rightarrow f^{(4)}(0) = -2 \cdot 3 \cdot 5^4$$

for $n \geq 1$: $f^{(n)}(0) = (-1)^{n-1} (n-1)! \cdot 5^n$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (n-1)! \cdot 5^n}{n!} x^n = \boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n-1} (5x)^n}{n}}$$

Radius of convergence?

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(5x)^{n+1} \cdot n}{(n+1)(5x)^n}\right|$$
$$= |5x| < 1$$
$$\boxed{R = \frac{1}{5}}$$

5). $f(x) = \sin\left(\frac{\pi x}{3}\right)$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{\left(\frac{\pi x}{3}\right)^{2n+1}}{(2n+1)!} \quad \text{for all real } x \quad (R=\infty)$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{\pi^{2n+1}}{3^{2n+1}(2n+1)!} \cdot x^{2n+1}$$

6). $f(x) = e^{-5x} = \sum_{n=0}^{\infty} \frac{(-5x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-5)^n}{n!} \cdot x^n$

7). $f(x) = x^4 - 4x^2 + 1$; $a=2$

$f(2) = 16 - 4\cdot 4 + 1 = 1$

$f'(x) = 4x^3 - 8x$; $f'(2) = 4\cdot 8 - 8\cdot 2 = 16$

$f''(x) = 12x^2 - 8$; $f''(2) = 12\cdot 4 - 8 = 40$

$f'''(x) = 24x$; $f^{(3)}(2) = 48$

$f^{(4)}(x) = 24$; $f^{(4)}(2) = 24$

$f^{(5)}(x) = 0$; $f^{(5)}(2) = 0$ $\quad f^{(n)}(x) = 0$ for $n \geq 5$

$$f(x) = 1 + 16(x-2) + \frac{40}{2!}(x-2)^2 + \frac{48}{3!}(x-2)^3 + \frac{24}{4!}(x-2)^4$$

$$= 1 + 16(x-2) + 20(x-2)^2 + 8(x-2)^3 + (x-2)^4$$

8). $f(x) = 2\cos(x)$; $a = 5\pi$

$$= 2 \cdot \sum_{n=0}^{\infty} (-1)^n \cdot \frac{(x-5\pi)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{2(-1)^n}{(2n)!}(x-5\pi)^{2n}$$

9). $f(x) = 4e^x + e^{6x}$

$$= 4\sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{(6x)^n}{n!} \qquad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(4+6^n)}{n!} x^n$$

10). Evaluate $5\int \frac{e^x - 1}{3x}\,dx$ as an infinite series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} \Rightarrow e^x - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!} \Rightarrow \frac{e^x - 1}{3x} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{3 \cdot n!}$$

$$\Rightarrow 5\int \frac{e^x - 1}{3x}\,dx = 5\sum_{n=1}^{\infty} \int \frac{x^{n-1}}{3 \cdot n!}\,dx = 5\sum_{n=1}^{\infty}\left(\frac{1}{3n!} \cdot \frac{x^n}{n}\right) + C$$

$$= C + \sum_{n=1}^{\infty} \frac{5x^n}{(3n) \cdot n!}$$

11). Same for $\int \frac{\cos(x) - 1}{x}\,dx$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!} = 1 + \sum_{n=1}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!}$$

$$\Rightarrow \cos(x) - 1 = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!} \Rightarrow \frac{\cos(x) - 1}{x} = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{x^{2n-1}}{(2n)!}$$

$$\Rightarrow \int \frac{\cos(x) - 1}{x}\,dx = \sum_{n=1}^{\infty}\left((-1)^n \cdot \frac{x^{2n}}{(2n) \cdot (2n)!}\right) + C$$

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