2.8 Worksheet: Related Rates

1. Let $A(t)$ be the area of a circle with radius $r(t)$, at time t in minutes. Suppose the radius is changing at the rate of $\frac{dr}{dt} = 3$ ft/min. Find the rate of change of the area at the moment in time when $r = 9$ ft.

$A(t) = \pi (r(t))^2$; $\frac{dr}{dt} = 3$ ft/min

Area of circle: $A = \pi r^2$

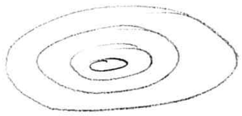
$\Downarrow$

$$\frac{dA}{dt} = \pi \cdot 2r(t) \cdot \frac{dr}{dt}$$

$$= \pi \cdot 2r(t) \cdot 3$$

$$\frac{dA}{dt} = 6\pi r(t) \Rightarrow \left.\frac{dA}{dt}\right|_{r=9} = 6\pi \cdot 9 = \boxed{54\pi}$$

2. A stone dropped into a still pond sends out a circular ripple whose radius increases at a constant rate of 4 ft/s. How rapidly is the area enclosed by the ripple increasing when the radius is 2 feet?



$A(t)$ = area enclosed @ time t

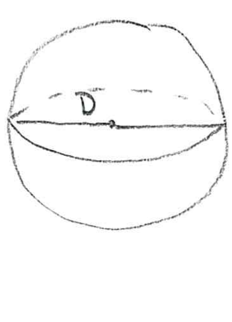
$r(t)$ = radius @ time t

$A(t) = \pi (r(t))^2$; $\frac{dr}{dt} = 4$

$\left.\frac{dA}{dt}\right|_{r=2} = ?$

$$\frac{dA}{dt} = 2\pi r(t) \cdot \underbrace{\frac{dr}{dt}}_{4} = 8\pi r(t) \Rightarrow \left.\frac{dA}{dt}\right|_{r=2} = \boxed{16\pi}$$

3. A spherical snowball is melting in such a way that its diameter is decreasing at rate of 2 cm/min. At what rate is the volume of the snowball decreasing when the diameter is 8 cm?



$V(t)$ = volume of snowball @ time t

$D(t)$ = diameter @ time t

$V(t) = \frac{\pi}{6} D(t)^3$; $\frac{dD}{dt} = -2$ (decreasing!)

$\left.\frac{dV}{dt}\right|_{D=8} = ?$

Volume of Sphere:

$V = \frac{4}{3}\pi r^3$

$= \frac{4}{3}\pi \left(\frac{D}{2}\right)^3$

$= \frac{1}{6}\pi \cdot D^3$

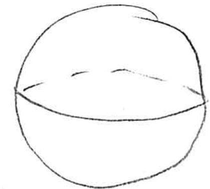
$$\frac{dV}{dt} = \frac{\pi}{2} D(t)^2 \underbrace{\frac{dD}{dt}}_{-2} = -\pi D(t)^2 \Rightarrow \left.\frac{dV}{dt}\right|_{D=8} = -64\pi$$

$\Rightarrow$ Decreasing at a rate of $\boxed{64\pi}$ cm^3/min

4. You are blowing air into a spherical balloon at a rate of 11 cubic inches per second. Given that the radius of the balloon is 5 inches when $t = 2$ seconds answer the following questions:

(a) How fast is the radius of the balloon growing at $t = 2$ seconds?

$V(t)$ = volume @ time t ; $r(t)$ = radius @ time t ;



$V(t) = \frac{4}{3}\pi r(t)^3$; $\frac{dV}{dt} = 11$ in³/s ; $r(2) = 5$ in.

$$\underbrace{\frac{dV}{dt}}_{=11} = \frac{4}{3}\pi \cdot 3 r(t)^2 \cdot \frac{dr}{dt} = 4\pi r(t)^2 \frac{dr}{dt} \Rightarrow 11 = 4\pi (5)^2 \left.\frac{dr}{dt}\right|_{t=2} \Rightarrow \left.\frac{dr}{dt}\right|_{t=2} = \boxed{\frac{11}{100\pi}} \text{ in/s}$$

(b) What is the rate of change of the surface area at $t = 2$ seconds?

Surface Area of Sphere:

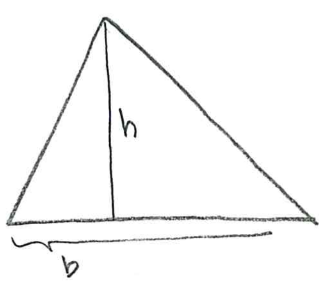
$A = 4\pi r^2$

$$A(t) = 4\pi r(t)^2$$

$$\frac{dA}{dt} = 4\pi \cdot 2r(t)\frac{dr}{dt} = 8\pi r(t)\frac{dr}{dt}$$

$$\Rightarrow \left.\frac{dA}{dt}\right|_{t=2} = 8\pi r(2)\left.\frac{dr}{dt}\right|_{t=2} = \cancel{8}\cancel{\pi}\cdot\cancel{5}\cdot\frac{11}{\cancel{100}\cancel{\pi}} = \boxed{\frac{22}{5}} \text{ in}^2/\text{s}$$

5. The height of a triangle is increasing at a rate of 1 cm/min while the area of the triangle is increasing at a rate of 6 square cm/min. At what rate is the base of the triangle changing when the height is 3 centimeters and the area is 9 square centimeters?



$h(t)$ = height ; $h'(t) = 1$

$A(t)$ = area ; $A'(t) = 6$

$b(t)$ = base ; $b'(t_0) = ?$ where t_0 = the time when $h(t_0) = 3$, $A(t_0) = 9$.

$9 = \frac{1}{2}b(t_0)\cdot 3$

$\Rightarrow b(t_0) = 6$

$$A(t) = \frac{1}{2}b(t)h(t)$$

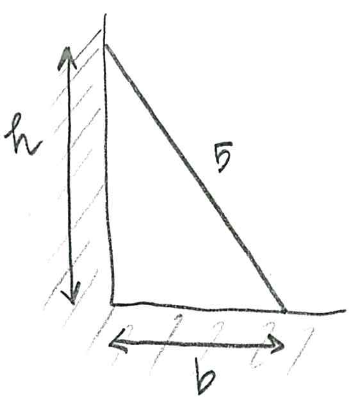
$$A'(t) = \frac{1}{2}b'(t)h(t) + \frac{1}{2}b(t)h'(t)$$

$$A'(t_0) = \frac{1}{2}b'(t_0)h(t_0) + \frac{1}{2}b(t_0)h'(t_0)$$

$$6 = \frac{1}{2}b'(t_0)\cdot 3 + \frac{1}{2}\cdot 6 \cdot 1$$

$$3 = \frac{1}{2}b'(t_0)\cdot 3 \Rightarrow \boxed{b'(t_0) = 2} \text{ cm/min}$$

6. The top of a 5 foot ladder, leaning against a vertical wall, is slipping down the wall at the rate of 2 feet per second. How fast is the bottom of the ladder sliding along the ground away from the wall when the bottom of the ladder is 3 feet away from the base of the wall?



$h(t)$ = height of ladder ; $h'(t) = -2$

$b(t)$ = dist. from bottom of ladder to wall ;

Q: $b'(t_0) = ?$ where t_0 = time when $b(t_0) = 3$

$$\boxed{h^2(t) + b^2(t) = 25}$$

$h^2(t_0) + 9 = 25$

$h(t_0) = 4$

$$2h(t)h'(t) + 2b(t)b'(t) = 0$$

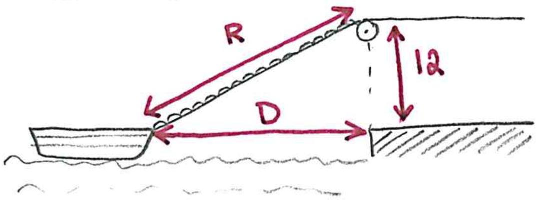
$$2h(t_0)h'(t_0) + 2b(t_0)b'(t_0) = 0$$

$$2\cdot 4\cdot(-2) + 2\cdot 3\cdot b'(t_0) = 0$$

$$6b'(t_0) = 16$$

$$b'(t_0) = \boxed{\frac{8}{3}} \text{ ft/s}$$

7. A boat is pulled into a dock by a rope attached to the bow (front end) of the boat and passing through a pulley on the dock that is 12 m higher than the bow of the boat. If the rope is pulled in at a rate of 3 m/s, at what speed is the boat approaching the dock when it is 5 m from the dock?

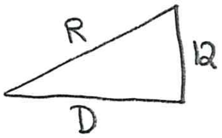


$R(t)$ = length of rope @ time t

$D(t)$ = distance from dock @ time t

$\frac{dR}{dt} = 3$

Q: $\left.\frac{dD}{dt}\right|_{D=5} = ?$



$\boxed{R^2 = 12^2 + D^2} \rightarrow \cancel{2}R\frac{dR}{dt} = \cancel{2}D\frac{dD}{dt}$

when $D=5$,

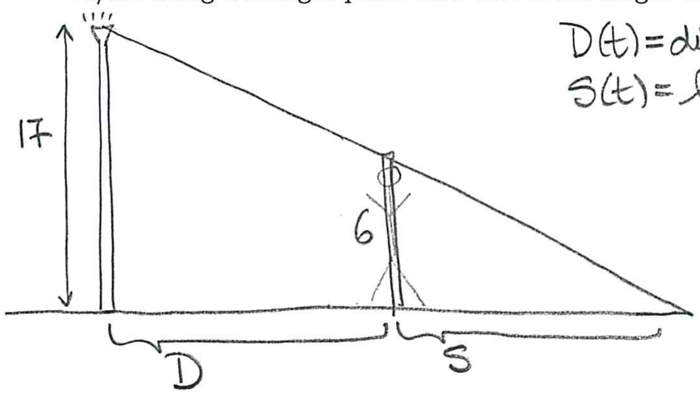
$R^2 = 144 + 25 = 13^2$

$\Rightarrow R = 13$

$\frac{dD}{dt} = \frac{R}{D} \cdot \frac{dR}{dt}$

$\left.\frac{dD}{dt}\right|_{D=5} = \frac{13}{5} \cdot 3 = \boxed{\frac{39}{5}}$ m/s

8. A street light is at the top of a pole that is 17 feet tall. A woman 6 ft tall walks away from the pole with a speed of 5 ft/sec along a straight path. How fast is the length of her shadow moving when she is 25 ft from the base of the pole?



$D(t)$ = distance from person & wall

$S(t)$ = length of shadow

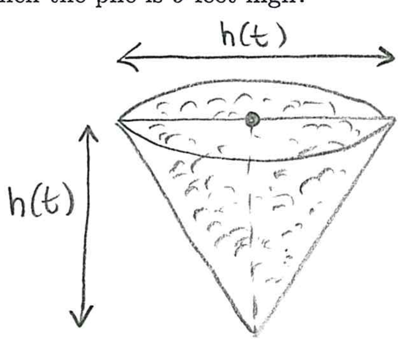
$\frac{dD}{dt} = 5$; $\left.\frac{dS}{dt}\right|_{D=25} = ?$

Similar Triangles:

$\frac{S}{6} = \frac{S+D}{17}$; $17S = 6S + 6D$; $\boxed{11S = 6D}$

$11S = 6D \Rightarrow 11\frac{dS}{dt} = 6\underbrace{\frac{dD}{dt}}_{30} \Rightarrow 11\frac{dS}{dt} = 30 \Rightarrow \frac{dS}{dt} = \boxed{\frac{30}{11}}$ ft/sec (constant)

9. Gravel is being dumped from a conveyor belt at a rate of 7 cubic feet per minute. It forms a pile in the shape of a right circular cone whose height and base diameter are always equal to each other. How fast is the height of the pile increasing when the pile is 9 feet high?



$V(t)$ = volume of cone; $\frac{dV}{dt} = 7$

$h(t)$ = base = height of cone.

$\left.\frac{dh}{dt}\right|_{h=9} = ?$

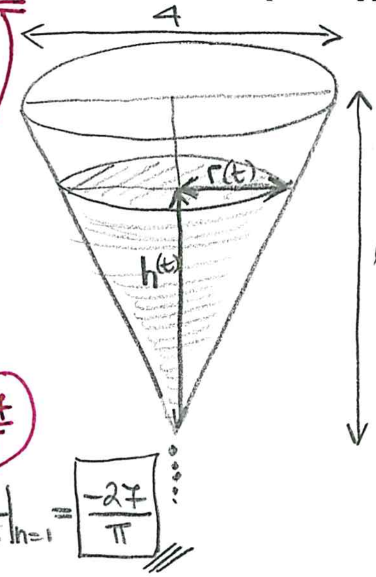
Volume of cone $= \pi r^2 \frac{h}{3}$ (r = radius of base)

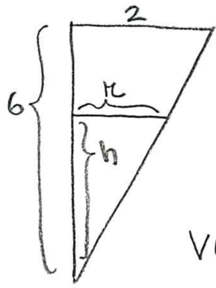
$V(t) = \pi\left(\frac{h(t)}{2}\right)^2 \frac{h(t)}{3} \Rightarrow \boxed{V(t) = \frac{\pi}{12} h(t)^3}$

$\Rightarrow \frac{dV}{dt} = \frac{\pi}{4} h(t)^2 \frac{dh}{dt} \Rightarrow 7 = \frac{\pi}{4} \cdot 9^2 \left.\frac{dh}{dt}\right|_{h=9}$

$\Rightarrow \left.\frac{dh}{dt}\right|_{h=9} = \boxed{\frac{28}{81\pi}}$ ft/min

10. A filter filled with liquid is in the shape of a vertex-down cone with a height of 6 inches and a diameter of 4 inches at its open (upper) end. If the liquid drips out the bottom of the filter at the constant rate of 3 cubic inches per second, how fast is the level of the liquid dropping when the liquid is 1 inches deep?





$V(t)$ = volume of liquid @ time t ; $\frac{dV}{dt} = -3$

$h(t)$ = height of liquid @ time t

$r(t)$ = radius of base liquid @ t ; $\left.\frac{dh}{dt}\right|_{h=1} = ?$

$$\boxed{V(t) = \pi r(t)^2 \frac{h(t)}{3}}$$

$$\frac{h}{r} = \frac{6}{2} \Rightarrow \boxed{h = 3r} \Rightarrow \boxed{r = \frac{h}{3}}$$

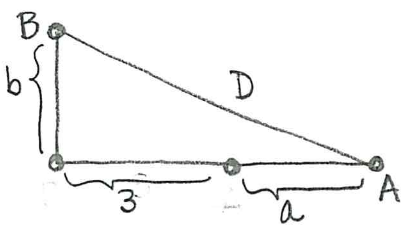
$$V(t) = \pi\left(\frac{h(t)}{3}\right)^2 \frac{h(t)}{3} \Rightarrow \boxed{V(t) = \pi\,\frac{h^3(t)}{27}}$$

$$\frac{dV}{dt} = \pi\,\frac{3h^2(t)\,h'(t)}{27} = \frac{\pi}{9}h^2(t)\,h'(t) \Rightarrow -3 = \frac{\pi}{9}(1)^2 \left.\frac{dh}{dt}\right|_{h=1}$$

$$\Rightarrow \left.\frac{dh}{dt}\right|_{h=1} = \boxed{\frac{-27}{\pi}}$$

$+\frac{27}{\pi}$

11. At noon, person A is 3 miles east of person B. Person A is walking east at 6 miles per hour and person B is walking north at 7 miles per hour. How fast is the distance between the people changing at 2 PM?



$a(t)$ = distance of A @ time t ; $\frac{da}{dt} = 6$ mph

$b(t)$ = distance of B @ time t ; $\frac{db}{dt} = 7$ mph

$D(t)$ = distance b/w A&B @ t ; Q: $\left.\frac{dD}{dt}\right|_{2PM} = ?$

$$D^2 = b^2 + (3+a)^2$$

$$2D(t)D'(t) = 2b(t)b'(t) + 2(3+a(t))\,a'(t)$$

$$\begin{aligned} a(2PM) &= 12 \\ b(2PM) &= 14 \end{aligned} \Rightarrow D^2(2PM) = 14^2 + 15^2 \Rightarrow D(2PM) = \sqrt{421}$$

$$2\sqrt{421}\;D'(2PM) = 2\cdot 14\cdot 7 + 2\cdot 15\cdot 6$$

$$\Rightarrow D'(2PM) = \boxed{\frac{188}{\sqrt{421}}} \text{ mph}$$