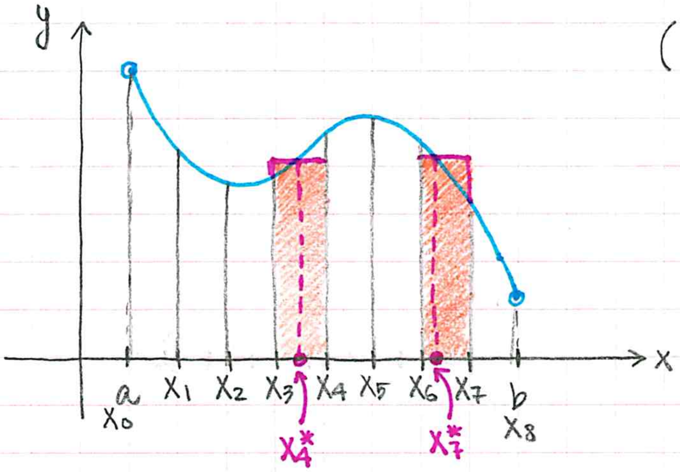
4.2. The Definite Integral

Riemann Sums: To estimate the area under the graph of $y=f(x)$ b/w $x=a$ & $x=b$:

- Divide the interval $[a,b]$ into n subintervals of width $\Delta x = \frac{b-a}{n}$.
- Let $x_0=a,\ x_1, x_2, \ldots, x_n=b$ be the endpoints of the intervals;
- In each subinterval $[x_{i-1}, x_i]$ choose a sample point x_i^*.

Provided the limit below exists:

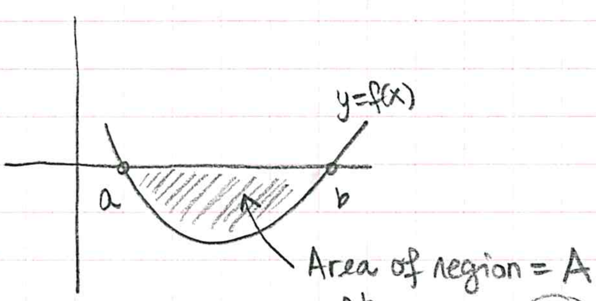
$$\boxed{\begin{aligned}\int_a^b f(x)\,dx &:= \sum_{i=1}^{\infty} f(x_i^*)\Delta x \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x\end{aligned}}$$



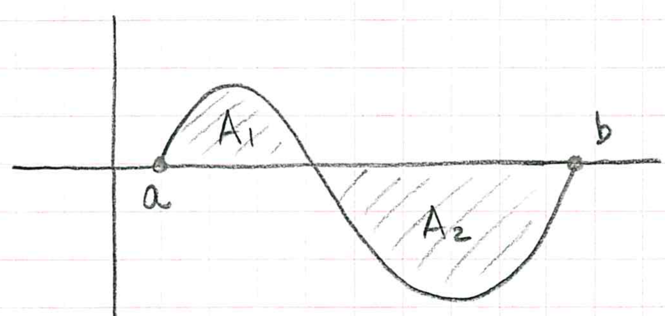
(The resulting limit must be the same regardless of the choice of sample pts. x_i^*).

Thm.: If f is continuous on $[a,b]$, or has only a finite number of jump discontinuities, then f is integrable on $[a,b]$ - i.e. $\int_a^b f(x)dx$ exists.

Signed Area: $\int_a^b f(x)\,dx$ is the "signed" area under the graph of $y=f(x)$, i.e. if f takes negative values, the integral computes "negative area":



$\Rightarrow \int_a^b f(x)dx = -A$



$\int_a^b f(x)\,dx = A_1 - A_2$

Properties of the definite integral:

$$\boxed{\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx}$$

$$\int_3^1 f(x)\,dx = -\int_1^3 f(x)\,dx.$$

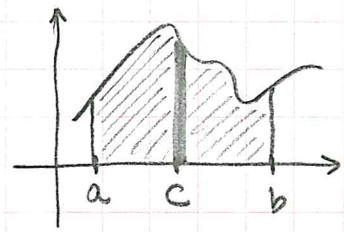
because, if $a>b$, the Δx changes from $\frac{b-a}{n}$ to $\frac{a-b}{n}$.

$$\boxed{\int_a^a f(x)\,dx = 0}$$

$$\boxed{\int_a^b (f(x)\pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx}$$

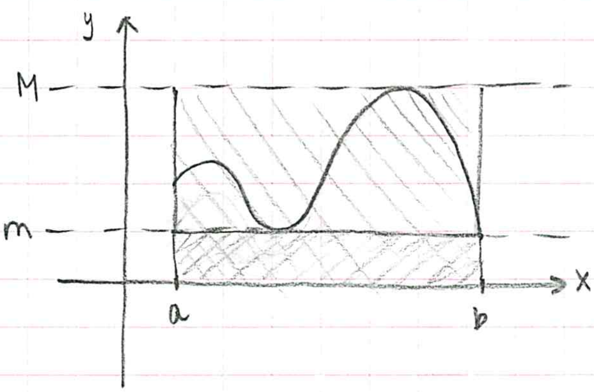
$$\boxed{\int_a^b c\cdot f(x)\,dx = c\cdot \int_a^b f(x)\,dx}$$

If $a<c<b$, then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

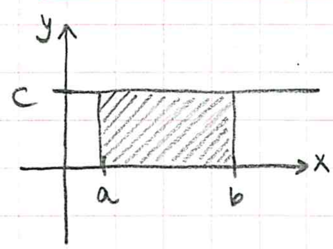


If $f(x)\geq g(x)$ for $a\leq x\leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

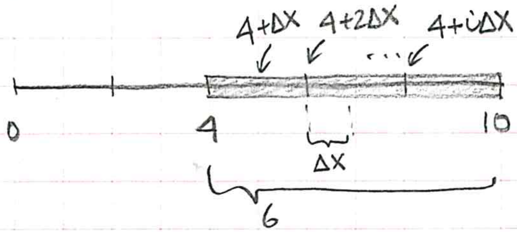
If $m\leq f(x)\leq M$ for $a\leq x\leq b$, then $m(b-a)\leq \int_a^b f(x)\,dx \leq M(b-a)$



because $\boxed{\int_a^b c\,dx = c(b-a)}$



1 $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{4+\frac{6i}{n}}\cdot\frac{6}{n} = \int_4^{10} f(x)\,dx.$ $f(x)=?$



$$\Delta x = \frac{10-4}{n} = \frac{6}{n}$$

$$f(4+i\Delta x) = \sqrt{4+\frac{6i}{n}} = \sqrt{4+i\cdot\frac{6}{n}}$$

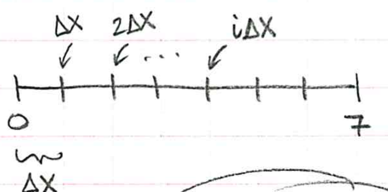
$$f(4+i\Delta x) = \sqrt{4+i\Delta x}$$

$$\Rightarrow f(x)=\sqrt{x}$$

2 $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{3+\frac{7i}{n}}\cdot\frac{7}{n} = \int_a^b f(x)\,dx.$

(Infinitely many choices here). (what is fixed is we should take $b-a=7$).

__Example:__ Take $a=0$, $b=7$, so $\Delta x = \frac{7}{n}$

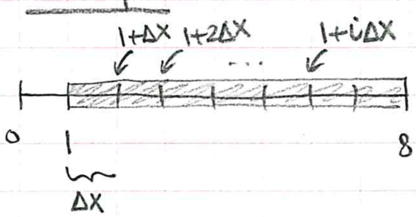


$$f(i\Delta x) = \frac{1}{3+\frac{7i}{n}} = \frac{1}{3+i\frac{7}{n}}$$

$$f(i\Delta x) = \frac{1}{3+i\Delta x} \Rightarrow f(x) = \frac{1}{3+x}$$

$$\Rightarrow \int_0^7 \frac{1}{3+x}\,dx$$

__Example:__ Take $a=1$, $b=8$, so $\Delta x = \frac{7}{n}$.



$$f(1+i\Delta x) = \frac{1}{3+\frac{7i}{n}} = \frac{1}{3+i\Delta x} = \frac{1}{2+1+i\Delta x}$$

$$f(1+i\Delta x) = \frac{1}{2+(1+i\Delta x)} \Rightarrow f(x) = \frac{1}{2+x}$$

$$\Rightarrow \int_1^8 \frac{1}{2+x}\,dx$$

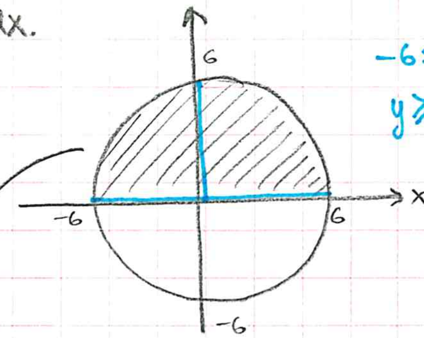
(3) Evaluate using areas: $\int_{-6}^{6} \sqrt{36-x^2}\,dx$.

$y = \sqrt{36-x^2} \Rightarrow y \geq 0$

$y^2 = 36 - x^2$

$$\boxed{x^2 + y^2 = 36}$$

Circle centered @ origin w/ radius 6.



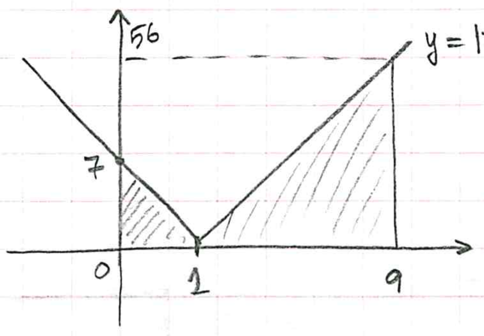
$-6 \leq x \leq 6$

$y \geq 0$

$\int_{-6}^{6} \sqrt{36-x^2}\,dx = 18\pi$

Area of half-circle: $\frac{1}{2}\cdot\pi r^2 = \frac{1}{2}\pi\cdot 36 = 18\pi$

(4) Same with $\int_0^9 |7x-7|\,dx$.



$$\int_0^9 |7x-7|\,dx = \int_0^1 |7x-7|\,dx + \int_1^9 |7x-7|\,dx$$

$$= \frac{1}{2}(1\cdot 7) + \frac{1}{2}(8\cdot 56)$$

$$= \frac{7}{2} + 4\cdot 56$$

(5) Given that $3 \leq f(x) \leq 4$ for $-7 \leq x \leq 7$, find the best possible estimation for $\int_{-7}^{7} f(x)\,dx$.

$$3\cdot 14 = 3(7-(-7)) \leq \int_{-7}^{7} f(x)\,dx \leq 4(7-(-7)) = 4\cdot 14$$