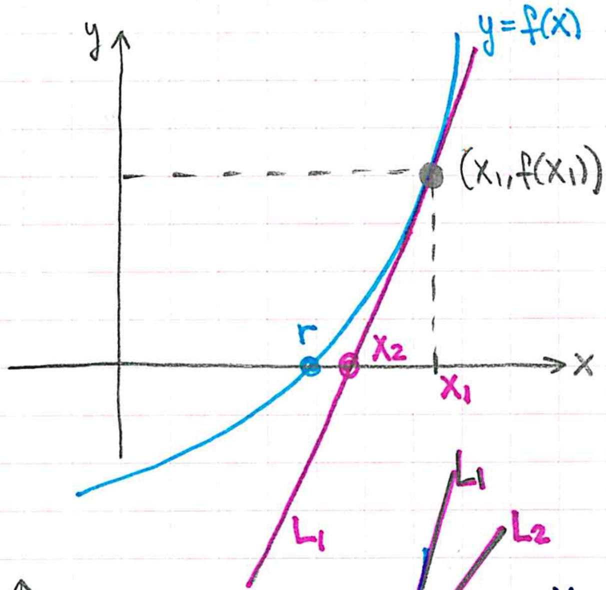
3.8. Newton's Method

Goal: Find numerical approximations of solutions to equations we cannot explicitly solve.

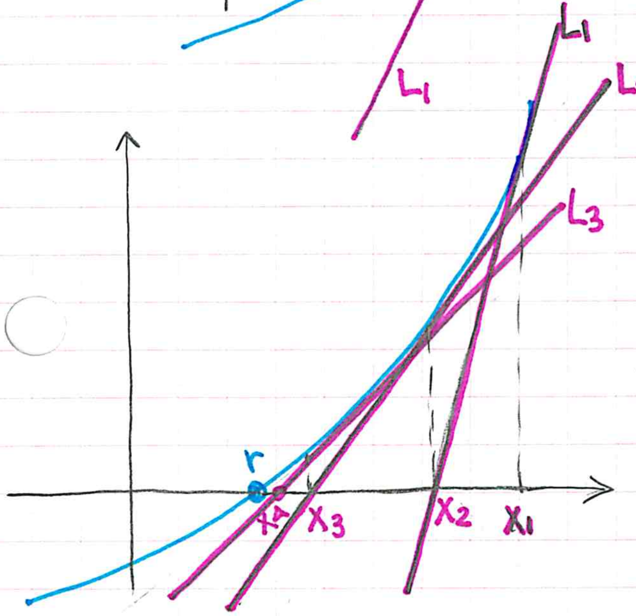


Example: Say we wish to approximate the x-intercept r of a function $f(x)$.

① Start w/ a guess x_1 (from a rough sketch, maybe from a computer generated graph etc).

② Consider the tangent line L_1 to the graph of f, at the pt. $(x_1, f(x_1))$

③ Look at x_2, the x-intercept of L_1:



Main Idea: The tangent line is "close" to the graph, so its x-intercept x_2 should be "close" to that of the graph (r).

Eqn. of L_1: $\quad y = f'(x_1)(x - x_1) + f(x_1)$

x-intercept: $\quad f'(x_1)(x - x_1) = -f(x_1)$

$$x - x_1 = -\frac{f(x_1)}{f'(x_1)}$$

$$\Rightarrow \boxed{x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}}$$

④ Repeat the procedure, now with x_2 in place of x_1 (the guess):

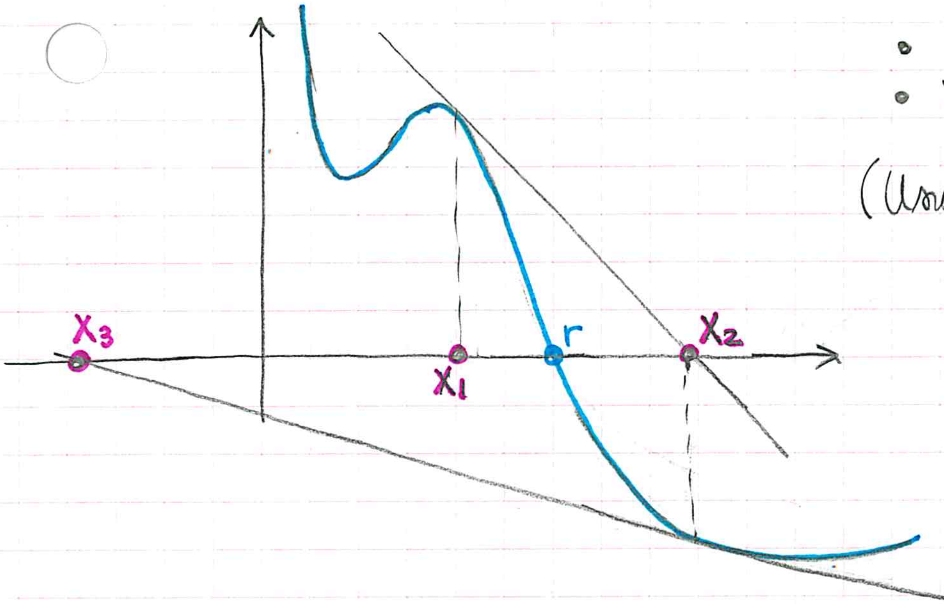
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

⋮

Generally: $\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$

IF the numbers x_n get closer & closer to r as $n \to \infty$, we say the sequence $\{x_n\}$ converges to r and write $\boxed{r = \lim_{n\to\infty} x_n}$

Remark: Sometimes Newton's method fails:



- X_2 is a worse approximation than X_1
- X_3 falls outside the domain!

(Usually a "better" X_1 must be chosen).

Example 1: $f(x) = x^3 + x - 4$. $\qquad f'(x) = 3x^2 + 1$

$$\left.\begin{array}{l} f(0) = -4 \\ f(2) = 6 \end{array}\right\} \Rightarrow$$ by IVT (f is continuous on $[0,2]$), f has a root in $(0,2)$.

Using $x_1 = 1$ as the starting value, find x_2 & x_3 (to 4 decimal places).

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{(1^3+1-4)}{(3 \cdot 1^2 + 1)} = 1 + \frac{2}{4} = 1 + \frac{1}{2} = 1.5$$

$x_2 = 1.5$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.5 - \frac{(1.5)^3 + (1.5) - 4}{3 \cdot (1.5)^2 + 1} = 1.3871$$

Example 2: Use Newton's method to approximate the positive value of X such that $X = 4.9\cos(X)$.

(Use $X_1 = 1$ as the starting point, and find X_2, X_3).

$$f(x) = X - 4.9\cos(X)$$
$$f'(x) = 1 + 4.9\sin(X)$$

$$X_2 = X_1 - \frac{f(X_1)}{f'(X_1)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1 - 4.9\cos(1)}{1 + 4.9\sin(1)} = 1.3216$$

$$X_3 = X_2 - \frac{f(X_2)}{f'(X_2)} = 1.3216 - \frac{1.3216 - 4.9\cos(1.3216)}{1 + 4.9\sin(1.3216)} = 1.3019$$

Example 3: Use Newton's Method to approximate a critical number of

$$f(x) = \frac{3}{8}X^8 + \frac{3}{5}X^5 + 4X + 5$$

near $X = 3$; use $X_1 = 3$ as the initial approx.

root of the first derivative!

Put $g(x) := f'(x) = 3X^7 + 3X^4 + 4$

$$g'(x) = 21X^6 + 12X^3$$

$$X_2 = X_1 - \frac{g(X_1)}{g'(X_1)} = 3 - \frac{3 \cdot 3^7 + 3 \cdot 3^4 + 4}{21 \cdot 3^6 + 12 \cdot 3^3} = 2.5645$$

$$X_3 = X_2 - \frac{g(X_2)}{g'(X_2)} = 2.5645 - \frac{3 \cdot (2.5645)^7 + 3 \cdot (2.5645)^4 + 4}{21 \cdot (2.5645)^6 + 12 \cdot (2.5645)^3}$$
$$= 2.1885$$