Quiz 7 - Solutions.

$$f(x) = \frac{x^3}{x^2-3}$$

① **Domain:** $(-\infty, -\sqrt{3}) \cup (-\sqrt{3}, \sqrt{3}) \cup (\sqrt{3}, \infty)$.

② **Monotonicity:**

$$f'(x) = \frac{3x^2(x^2-3) - x^3(2x)}{(x^2-3)^2} = \frac{3x^4 - 9x^2 - 2x^4}{(x^2-3)^2}$$

$$= \frac{x^4 - 9x^2}{(x^2-3)^2} = \frac{x^2(x^2-9)}{(x^2-3)^2}$$

always (+) $\frac{x^2}{(x^2-3)^2}$ (x^2-9) → parabola, -3, 3

x	-3	$-\sqrt{3}$	$\sqrt{3}$	3
$f'(x)$	$+\ 0\ -$	$\|\ -\ \|$	$-\ 0\ +$	$+$
	↗	↘	↘	↗

$f \uparrow$ on $(-\infty, -3) \cup (3, \infty)$

$f \downarrow$ on $(-3, -\sqrt{3}) \cup (-\sqrt{3}, \sqrt{3}) \cup (\sqrt{3}, 3)$.

③ **Concavity:**

$$f''(x) = \frac{(4x^3 - 18x)(x^2-3)^2 - (x^4-9x^2)\cdot 2(x^2-3)\cdot 2x}{(x^2-3)^4}$$

$$= \frac{(4x^3-18x)(x^2-3) - 4x(x^4-9x^2)}{(x^2-3)^3}$$

$$= \frac{6x(x^2+9)}{(x^2-3)^3}$$

$4x^5 - 12x^3 - 18x^3 + 54x$
$- 4x^5 + 36x^3$
$= 6x^3 + 54x$
$= 6x(x^2+9)$

x		$-\sqrt{3}$		0		$\sqrt{3}$	
$6x(x^2+9)$	$-\ -\ -\ -$		0	$+$	$+\ +\ +$		
$(x^2-3)^3$	$+\ +$	0	$-\ -\ -$		0	$+\ +$	
f''	$-$	$\|$	$+$	0	$-$	$\|$	$+$
	∩		∪		∩		∪

f is ∩ on $(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$

f is ∪ on $(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$

④ Vertical Asymptotes: $x=\sqrt{3}$, $x=-\sqrt{3}$

$$f(x)=\frac{x^3}{x^2-3}$$

$$\lim_{x\to-\sqrt{3}^-} f(x)=\frac{\ominus}{0_+}=-\infty \qquad \lim_{x\to-\sqrt{3}^+} f(x)=\frac{\ominus}{0_-}=+\infty$$

$$\lim_{x\to\sqrt{3}^-} f(x)=\frac{\oplus}{0_-}=-\infty \qquad \lim_{x\to\sqrt{3}^+} f(x)=\frac{\oplus}{0_+}=+\infty,$$

Horizontal Asymptotes / Limits at ∞:

$$\lim_{x\to\infty}\frac{x^3}{x^2-3}=\lim_{x\to\infty}\frac{x}{1-3/x^2}=(+\infty) \qquad \lim_{x\to-\infty}\frac{x^3}{x^2-3}=\lim_{x\to-\infty}\frac{-x}{1-3/x^2}=(-\infty)$$

(no horizontal asymptotes).

Slant Asymptotes:

$$x^3=x(x^2-3)+3x \Rightarrow f(x)=\frac{x(x^2-3)+3x}{x^2-3}=x+\left(\frac{3x}{x^2-3}\right)$$

$\frac{3x}{x^2-3} \to 0$ as $x\to\infty$ and $x\to-\infty$

$$\boxed{y=x}$$

⑤

x	$-\infty$		-3		$-\sqrt{3}$		0		$\sqrt{3}$		3		∞
$f'(x)$	+	+ +	0	− −		− − −		− −	0	+	+	+	+
$f''(x)$	−	− − − − −				+	0	−		+	+	+	+
$f(x)$	$-\infty$	↗	$-9/2$	↘	$-\infty$	∩	$+\infty$ ↘ 0	∪	$-\infty$	↘	$9/2$	∪ ↗	$+\infty$