① $f(x) = \dfrac{3x}{x-1}$

Domain: $(x \neq 1)$ $\quad (-\infty, 1) \cup (1, \infty)$.

First Derivative: $\quad f'(x) = \dfrac{3(x-1) - 3x \cdot 1}{(x-1)^2} = \dfrac{-3}{(x-1)^2} < 0$ for all $x \neq 1$

Second Derivative: $\quad f'(x) = -3(x-1)^{-2}$

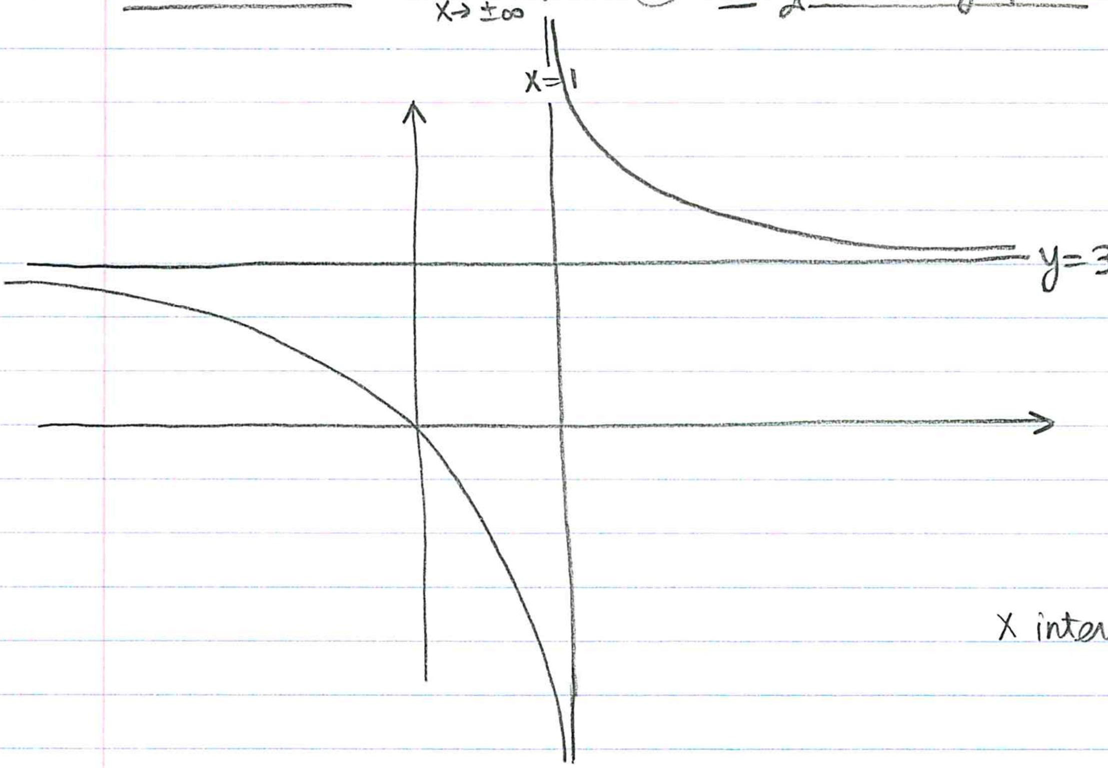
$$f''(x) = 6(x-1)^{-3} = \frac{6}{(x-1)^3}$$

↳ changes sign @ $x=1$

x	$-\infty$	1	∞
$f'(x)$	$- - - -$		$- - - - - - -$
$f''(x)$	$- - - -$		$+ + + + + +$
$f(x)$	$3 \searrow \frown$ $-\infty$		$+\infty \searrow \smile$ 3

Vertical Asymptote at $x=1$; Side Limit:
$$\begin{cases} \lim\limits_{x\to 1^-} f(x) = \dfrac{3}{0_-} = -\infty \\ \lim\limits_{x\to 1^+} f(x) = \dfrac{3}{0_+} = +\infty \end{cases}$$

Limit at $\pm\infty$: $\quad \lim\limits_{x\to\pm\infty} f(x) = 3$ (Horizontal Asymptote)



x intercept? $3x=0$; $x=0$

② $f(x) = \dfrac{2x^2}{x^2-1}$

Domain: $x^2 \neq 1$ $\quad (-\infty,-1) \cup (-1,1) \cup (1,\infty)$.

First Derivative:

$$f'(x) = \frac{4x(x^2-1) - 2x^2 \cdot 2x}{(x^2-1)^2} = \frac{\cancel{4x^3} - 4x - \cancel{4x^3}}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}$$

Critical point: $x = 0$

Second Derivative:

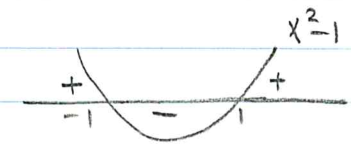
$$f''(x) = \frac{-4(x^2-1)^{\cancel{2}} + 4x \cdot 2\cancel{(x^2-1)} \cdot 2x}{(x^2-1)^{\cancel{4}\,3}}$$

$$= \frac{-4(x^2-1) + 16x^2}{(x^2-1)^3} = \frac{12x^2+4}{(x^2-1)^3}$$

local max

x	$-\infty$ … -1	-1 … 0 … 1	1 … ∞
$f'(x)$	+ + +	+ 0 −	− − − −
$f''(x)$	+ + +	− − −	+ + + +
$f(x)$	$2 \nearrow +\infty$, ∪	$-\infty \nearrow 0 \searrow -\infty$, ∩	$+\infty \searrow 2$, ∪

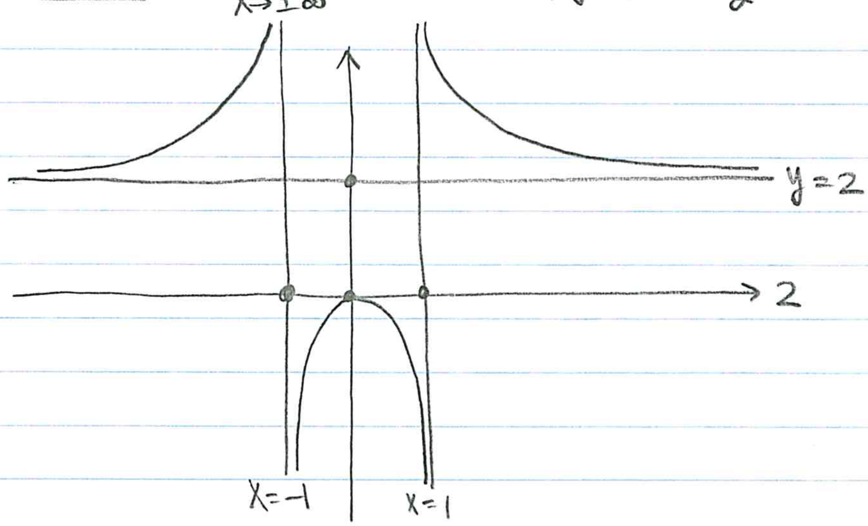
Vertical Asymptotes: $x=-1$, $x=1$



$$\lim_{x\to -1^-} f(x) = \frac{2}{0_+} = +\infty \qquad \lim_{x\to 1^-} f(x) = \frac{2}{0_-} = -\infty$$

$$\lim_{x\to -1^+} f(x) = \frac{2}{0_-} = -\infty \qquad \lim_{x\to 1^+} f(x) = \frac{2}{0+} = +\infty$$

Limits at $\pm\infty$: $\displaystyle\lim_{x\to\pm\infty} f(x) = 2$ ($y=2$ Horizontal Asymptote).



③ $f(x) = \dfrac{x^2}{\sqrt{x+1}}$

Domain: $x > -1$ $\quad (-1, \infty)$.

First Derivative: $f'(x) = \dfrac{2x\sqrt{x+1} - x^2 \cdot \frac{1}{2\sqrt{x+1}}}{x+1} \quad \Big| \times \dfrac{2\sqrt{x+1}}{2\sqrt{x+1}}$

$$= \frac{4x(x+1) - x^2}{2(x+1)\sqrt{x+1}} = \frac{3x^2+4x}{2(x+1)\sqrt{x+1}} = \frac{\underset{\uparrow\, 0}{x}\,\underset{\uparrow\, -4/3}{(3x+4)}}{2(x+1)^{3/2}}$$

Second Derivative:

$$f''(x) = \frac{(6x+4)(x+1)^{3/2} - (3x^2+4x)\cdot \frac{3}{2}(x+1)^{1/2}}{2(x+1)^3} \quad \Big| \times \frac{2/(x+1)^{1/2}}{2/(x+1)^{1/2}}$$

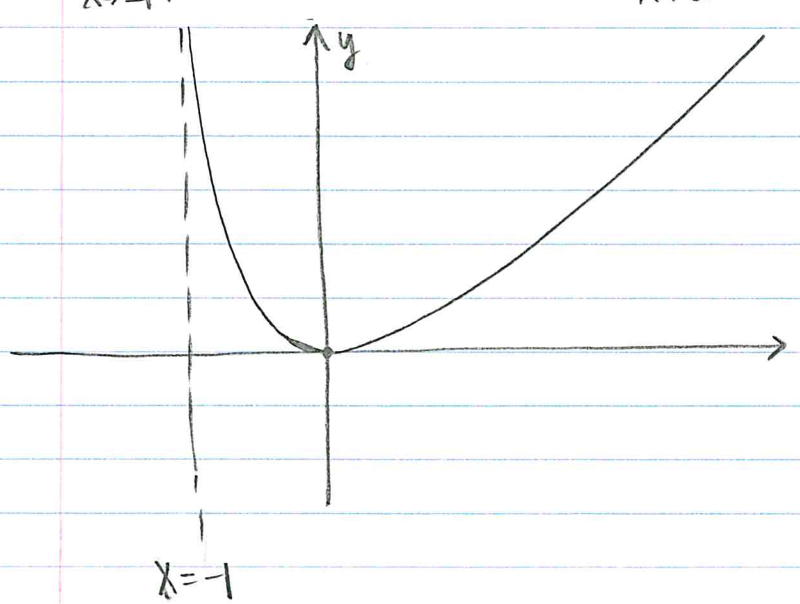
$$= \frac{2(6x+4)(x+1) - 3(3x^2+4x)}{4(x+1)^{5/2}}$$

$$= \frac{12x^2+20x+8-9x^2-12x}{4(x+1)^{5/2}} = \frac{3x^2+8x+8}{4(x+1)^{5/2}}$$

$\Delta = 64 - 96 < 0$
No real roots!
(always ⊕)

x	-1			0					∞
$f'(x)$		−	−	− 0	+	+	+	+	+
$f''(x)$		+	+	+	+	+	+	+	+
$f(x)$	$+\infty$	↘		0	↗				∞

$$\lim_{x\to -1^+} f(x) = \frac{1}{0_+} = +\infty \qquad \lim_{x\to\infty} f(x) = \infty$$



④ $f(x) = \dfrac{x^3}{x^2+1}$

Domain: $\mathbb{R}$

First Derivative: $f'(x) = \dfrac{3x^2(x^2+1) - x^3 \cdot 2x}{(x^2+1)^2} = \dfrac{3x^4+3x^2-2x^4}{(x^2+1)^2} = \dfrac{x^4+3x^2}{(x^2+1)^2}$

$= \dfrac{x^2(x^2+3)}{(x^2+1)^2}$ $\qquad x=0$ critical pt.

Second Derivative:

$$f''(x) = \frac{(4x^3+6x)(x^2+1)^{\cancel{2}} - (x^4+3x^2)\cdot 2\cancel{(x^2+1)}\cdot 2x}{(x^2+1)^{\cancel{4}\,3}}$$

$$= \frac{\cancel{4x^5}+6x^3+4x^3+6x-\cancel{4x^5}-12x^3}{(x^2+1)^3} = \frac{-2x^3+6x}{(x^2+1)^3}$$

$$= \frac{2x(3-x^2)}{(x^2+1)^3}$$

x	$-\sqrt{3}$ $\quad 0 \quad$ $\sqrt{3}$
$2x$	— — — 0 + +
$3-x^2$	— — 0 + + 0 — —
f''	+ 0 — 0 + 0 —

inflection pts.

x	$-\sqrt{3}$ $\quad 0 \quad$ $\sqrt{3}$
$f'(x)$	+ + + + 0 + + + +
$f''(x)$	+ + 0 — 0 + 0 — — —
$f(x)$	$-\infty$ ↗ ∪ $\frac{-3\sqrt{3}}{4}$ ∩ ↗ 0 ↗ ∪ $\frac{3\sqrt{3}}{4}$ ↗ ∩ ∞

$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \dfrac{x}{1+\frac{1}{x^2}} = -\infty$ $\qquad \lim_{x\to\infty} f(x) = \infty$

Slant Asymptote:

$x^3 = x(x^2+1) - x$

$f(x) = \dfrac{x(x^2+1)-x}{x^2+1} = x - \dfrac{x}{x^2+1}$

$y = x$