Final: Worksheet 3

1) Find K such that f is continuous everywhere:

$$f(x) = \begin{cases} x^2+2x, & \text{if } x \le K \\ 2x^2+2x-1, & \text{if } x > K \end{cases}$$

$$\lim_{x\to K^-} f(x) = \lim_{x \to K^+} f(x)$$

(f is continuous everywhere except at $x=K$)

$$K^2+2K = 2K^2+2K-1$$
$$0 = K^2-1$$
$$0 = (K-1)(K+1) \Rightarrow K=-1 \text{ or } K=1$$

2) Same for

$$f(x) = \begin{cases} x^2+x-2K, & \text{if } x \le 2 \\ 6x+K, & \text{if } x > 2 \end{cases}$$

$$\lim_{x\to 2^-} f(x) = \lim_{x \to 2^+} f(x)$$

$$2^2+2-2K = 6\cdot 2+K$$
$$6-2K = 12+K$$
$$-6 = 3K \Rightarrow K=-2$$

3) Find K such that $y = \underline{\underline{-4}}x+K$ is a tangent line to the graph of $f(x) = x^3+3x^2-x+1$.

Eqn. of tangent line at $(x_0, f(x_0))$: $y - f(x_0) = \underline{\underline{f'(x_0)}}\,(x-x_0)$

Slope $= -4 \Rightarrow$ find x_0 s.t. $f'(x_0) = -4$.

$$f'(x) = 3x^2+6x-1 \Rightarrow 3x_0^2+6x_0-1 = -4$$
$$3x_0^2+6x_0+3 = 0$$
$$x_0^2+2x_0+1 = 0$$
$$(x_0+1)^2 = 0 \Rightarrow x_0 = -1$$

$$f(x_0) = f(-1) = -1+3+1+1 = 4$$

$\Rightarrow$ Tangent line: (with $x_0=-1$)

$$y-4 = -4(x+1)$$
$$y-4 = -4x-4$$
$$y = -4x \Rightarrow \boxed{K = 0}$$

④ Find k such that $y = 2x+6$ is a tangent line to the graph of $f(x) = x^2 - 2x + k$ at the point where $x = 2$.

Eqn. of tangent line (at $x=2$): $y - f(2) = f'(2)(x-2)$

$$\left.\begin{array}{l} f'(x) = 2x-2 \Rightarrow f'(2) = 2 \\ f(2) = 4-4+k = \underline{\underline{k}} \end{array}\right\} \Rightarrow y - k = 2(x-2)$$

$$\left.\begin{array}{l} y = 2x - 4 + k \\ \text{supposed to be } y = 2x+6 \end{array}\right\} \Rightarrow$$

$$\cancel{2x} - 4 + k = \cancel{2x} + 6$$

$$-4 + k = 6$$

$$\boxed{k = 10}$$

⑤ Find k such that $\int_k^{k+1} (2x+1)\,dx = 2$

$$\int_k^{k+1} (2x+1)\,dx = (x^2+x)\Big|_k^{k+1} = \left((k+1)^2 + (k+1)\right) - (k^2+k)$$

$$= \cancel{k^2} + 2k + 1 + \cancel{k} + 1 - \cancel{k^2} - \cancel{k}$$

$$= 2k + 2$$

$$2k+2 = 2 \Rightarrow 2k = 0 \Rightarrow k = 0$$

⑥ Find k such that the average value of $f(x) = \frac{1}{\sqrt{x}} + 2x - k$ on the interval $[0, k]$ is equal to 8.

$$f_{avg} = \frac{1}{k}\int_0^k \left(\frac{1}{\sqrt{x}} + 2x - k\right)dx = \frac{1}{k}\left(2\sqrt{x} + x^2 - kx\right)\Big|_{x=0}^{x=k}$$

$$= \frac{1}{k}\left(\left(2\sqrt{k} + \cancel{k^2} - \cancel{k^2}\right) - (0+0-0)\right) = \frac{1}{k}\cdot 2\sqrt{k} = \frac{2}{\sqrt{k}}$$

$$\frac{2}{\sqrt{k}} = 8 \Rightarrow \frac{1}{4} = \sqrt{k} \Rightarrow k = \frac{1}{16}$$

7. Find K such that

$$\int_1^2 \frac{(x+K)(x-K)}{2x^2}\,dx = 0$$

$$\int_1^2 \frac{(x+K)(x-K)}{2x^2}\,dx = \int_1^2 \frac{x^2-K^2}{2x^2}\,dx = \int_1^2 \left(\frac{1}{2} - \frac{K^2}{2x^2}\right)dx$$

$$= \left(\frac{1}{2}x + \frac{K^2}{2}\frac{1}{x}\right)\Bigg|_{x=1}^{x=2}$$

$$= \left(\frac{1}{2}\cdot 2 + \frac{K^2}{2}\cdot\frac{1}{2}\right) - \left(\frac{1}{2} + \frac{K^2}{2}\right)$$

$$= 1 + \frac{K^2}{4} - \frac{1}{2} - \frac{K^2}{2}$$

$$= \frac{1}{2} - \frac{K^2}{4} = \frac{2-K^2}{4} \Rightarrow \frac{2-K^2}{4} = 0 \Rightarrow 2-K^2=0 \Rightarrow K^2=2$$

$$\Rightarrow \boxed{K = -\sqrt{2} \text{ or } \sqrt{2}}$$

8. Find K such that

$$\int_{-K}^{K} \frac{(x-1)(x+3)}{x^4}\,dx = 0$$

$$\int_{-K}^{K} \frac{(x-1)(x+3)}{x^4}\,dx = \int_{-K}^{K} \frac{x^2+2x-3}{x^4}\,dx = \int_{-K}^{K}\left(\frac{1}{x^2} + \frac{2}{x^3} - \frac{3}{x^4}\right)dx$$

$$= \left(\frac{-1}{x} - \frac{1}{x^2} + \frac{1}{x^3}\right)\Bigg|_{x=-K}^{x=K} = \left(\frac{-1}{K} - \frac{1}{K^2} + \frac{1}{K^3}\right) - \left(\frac{1}{K} - \frac{1}{K^2} - \frac{1}{K^3}\right)$$

$$= \frac{-1}{K} - \cancel{\frac{1}{K^2}} + \frac{1}{K^3} - \frac{1}{K} + \cancel{\frac{1}{K^2}} + \frac{1}{K^3}$$

$$= \frac{2}{K^3} - \frac{2}{K} = \frac{2-2K^2}{K^3}$$

$$\frac{2-2K^2}{K^3} = 0 \Rightarrow 2-2K^2 = 0 \Rightarrow K^2 = 1 \Rightarrow \boxed{K=1 \text{ or } K=-1}$$

(9) Find all values c which satisfy MVT for

$$f(x) = x^3 + 2x^2 - x \text{ on } [-1, 2].$$

$$f'(c) = \frac{f(2) - f(-1)}{2-(-1)} = \frac{(8+8-2)-(\cancel{-1}+2\cancel{+1})}{3} = \frac{12}{3} = 4$$

$$f'(x) = 3x^2 + 4x - 1$$

$$f'(c) = 4 \Rightarrow \quad 3c^2 + 4c - 1 = 4$$
$$3c^2 + 4c - 5 = 0$$

$$c = \frac{-4 \pm \sqrt{16+60}}{6} = \frac{-4 \pm \sqrt{76}}{6}$$

Are they both in $[-1, 2]$?

$\frac{-4-\sqrt{76}}{6}$ is too small (<-1)

$$\frac{-4-\sqrt{76}}{6} < \frac{-4-\sqrt{64}}{6} = -2$$

$$\frac{-4+\sqrt{76}}{6} < \frac{-4+\sqrt{81}}{6} = \frac{5}{6} < 2$$

$$\Rightarrow \text{only } \boxed{c = \frac{-4+\sqrt{76}}{6}}$$

(10) Let f be a function w/ first derivative $f'(x) = \frac{2x^2-5}{x^2}$ for $x>0$. Given that $f(1) = 7$ and $f(5) = 11$, what value $c \in (1,5)$ satisfies the conclusion of MVT for f on the interval $[1,5]$?

$$\frac{2c^2-5}{c^2} = \frac{11-7}{5-1} \Rightarrow \frac{2c^2-5}{c^2} = \frac{4}{4} = 1 \Rightarrow 2c^2 - 5 = c^2$$

$$\Rightarrow c^2 = 5 \Rightarrow c = \pm\sqrt{5}$$

$$\Rightarrow c = \sqrt{5} \quad (\text{b/c } -\sqrt{5} \notin (1,5)).$$

(11) f is continuous on $[0,2]$ and has values:

x	0	1	2
$f(x)$	1	k	2

The eqn. $f(x) = 1/2$ must have at least 2 solutions in the interval $[0,2]$ if k

a). 0 b). $\frac{1}{2}$ c). 1 d). 2 e). 3

IVT

(12) Selected values of a continuous function g:

x	0	2	5	9	11
$g(x)$	1	2.8	1.7	1	3.4

① ② ③

For $0 \le x \le 11$, what is the minimum number of times $g(x) = 2$?

(IVT) a). One b). Two c). Three d). Four e). Five