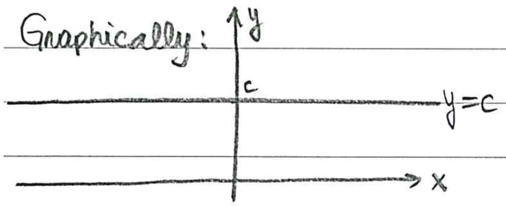
2.3 Differentiation Formulas

* **Derivative of a constant function:** $\boxed{\frac{d}{dx}(c) = 0}$

Graphically:



$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad (f(x) = c)$$

$$= \lim_{h \to 0} \frac{c - c}{h} = \boxed{0}$$

* **Power Functions:** $\boxed{\frac{d}{dx}(x^n) = n x^{n-1}}$ for any real number n

$$\frac{d}{dx}(x^3) = 3x^2; \quad \frac{d}{dy}(y^{-3}) = -3y^{-4}; \quad \frac{d}{da}(a^{1/2}) = \frac{1}{2} a^{-1/2}$$

* **Combining functions:**
  * **Multiplying by a constant:** $\frac{d}{dx}[cf(x)] = c \frac{d}{dx} f(x)$ $\qquad (cf)' = cf'$
  * **Sum & Difference Rule:** $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x).$ $\qquad (f \pm g)' = f' \pm g'$
  * **PRODUCT RULE:** $\boxed{\frac{d}{dx}[f(x) g(x)] = \frac{df}{dx}(x) \cdot g(x) + f(x) \cdot \frac{dg}{dx}(x)}$

$$\boxed{(fg)' = f'g + fg'}$$

  * **QUOTIENT RULE:** $\boxed{\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\frac{df}{dx}(x) \cdot g(x) - f(x) \cdot \frac{dg}{dx}(x)}{g^2(x)}}$

$$\boxed{\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}}$$

2.4. Derivatives of Trigonometric Functions

Fundamental Limit: $\boxed{\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}=1}$

$\Rightarrow \boxed{\lim_{\theta\to 0}\frac{\cos(\theta)-1}{\theta}=0}$

$$\lim_{\theta\to 0}\frac{\cos(\theta)-1}{\theta}\cdot\frac{\cos(\theta)+1}{\cos(\theta)+1}=\lim_{\theta\to 0}\frac{\cos^2(\theta)-1}{\theta(\cos(\theta)+1)}=\lim_{\theta\to 0}\frac{-\sin^2\theta}{\theta(\cos\theta+1)}$$

$$=\lim_{\theta\to 0}-\underbrace{\frac{\sin\theta}{\theta}}_{\downarrow\theta\to 0\ 1}\cdot\underbrace{\frac{\sin\theta}{\cos\theta+1}}_{\downarrow\theta\to 0\ 0}=1\cdot 0=\boxed{0}.$$

$\Rightarrow \boxed{\frac{d}{dx}(\sin x)=\cos(x)}$

$$\lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}=\lim_{h\to 0}\frac{\sin x\cosh+\cos x\sin h-\sin x}{h}$$

$$=\lim_{h\to 0}\left(\frac{\sin x\cosh-\sin x}{h}+\frac{\cos x\sin h}{h}\right)$$

$$=\lim_{h\to 0}\left(\sin x\cdot\underbrace{\frac{\cosh-1}{h}}_{\downarrow h\to 0\ 0}+\cos x\cdot\underbrace{\frac{\sin h}{h}}_{\downarrow h\to 0\ 1}\right)=\underline{\underline{\cos(x)}}.$$

Similarly: $\boxed{\frac{d}{dx}(\cos x)=-\sin(x)}$

Examples:

1). $y=x^{1/3}\sin(x)\cos(x)$; $\quad \frac{dy}{dx}=\frac{1}{3}x^{-2/3}\sin(x)\cos(x)+x^{1/3}\cos^2(x)-x^{1/3}\sin^2(x).$

2). $f(x)=\frac{\sin(x)}{\cos(x)}$; $\quad f'(x)=\frac{(\sin x)'\cos x-\sin x(\cos x)'}{\cos^2(x)}=\frac{\cos^2x+\sin^2x}{\cos^2x}=\frac{1}{\cos^2x}$

$\boxed{\frac{d}{dx}(\tan x)=\sec^2x}$

3). $f(\ell)=\frac{\tan(\ell)}{2\ell^5-\ell^{1/3}}$; $\quad f'(\ell)=\frac{\sec^2(\ell)(2\ell^5-\ell^{1/3})-\tan(\ell)(10\ell^4-\frac{1}{3}\ell^{-2/3})}{(2\ell^5-\ell^{1/3})^2}.$

Examples with $\boxed{\lim\limits_{☺ \to 0} \frac{\sin ☺}{☺} = 1.}$

1). $\lim\limits_{x\to 0} \frac{\sin(2x)}{7x} = \lim\limits_{x\to 0} \underbrace{\left(\frac{\sin(2x)}{2x}\right)}_{\downarrow x\to 0} \cdot \frac{1}{7/2} = \frac{1}{7/2} = \boxed{\frac{2}{7}}$

2). $\lim\limits_{x\to 0} \frac{\tan(x)}{6x} = \lim\limits_{x\to 0} \frac{\sin(x)}{\cos(x)} \frac{1}{6x} = \lim\limits_{x\to 0} \underbrace{\left(\frac{\sin(x)}{x}\right)}_{\substack{\downarrow x\to 0 \\ 1}} \cdot \underbrace{\left(\frac{1}{6\cos(x)}\right)}_{\substack{\downarrow x\to 0 \\ 1/6}} = \boxed{\frac{1}{6}}$

3). $\lim\limits_{t\to 64} \frac{\sin(\sqrt{t}-8)}{t-64} = \lim\limits_{t\to 64} \frac{\sin(\sqrt{t}-8)}{(\sqrt{t}-8)(\sqrt{t}+8)} = \lim\limits_{t\to 64} \underbrace{\left(\frac{\sin(\sqrt{t}-8)}{(\sqrt{t}-8)}\right)}_{\substack{\downarrow t\to 64 \\ 1}} \underbrace{\left(\frac{1}{\sqrt{t}+8}\right)}_{\substack{\downarrow t\to 64 \\ 1/16}} = \boxed{\frac{1}{16}}$

4). $\lim\limits_{s\to 0} \frac{\sin(s^2-9s)}{s} = \lim\limits_{s\to 0} \left(\underbrace{\frac{\sin(s^2-9s)}{s^2-9s}}_{\substack{\downarrow s\to 0 \\ 1}} \underbrace{(s-9)}_{\substack{\downarrow s\to 0 \\ -9}} \right) = \boxed{-9}$

5). $\lim\limits_{t\to 0} \frac{\sin(6\sin t)}{t} = \lim\limits_{t\to 0} \underbrace{\left(\frac{\sin(6\sin t)}{(6\sin t)}\right)}_{\substack{\downarrow t\to 0 \\ 1}} \cdot \underbrace{\left(\frac{6\sin t}{t}\right)}_{\substack{\downarrow t\to 0 \\ 6}} = \boxed{6}$

6). $\lim\limits_{y\to 0+} \frac{\sin(6y)}{\sin(\sqrt{y})} = \lim\limits_{y\to 0+} \frac{\left(\frac{\sin(6y)}{6y}\right)\cdot(6y)}{\left(\frac{\sin(\sqrt{y})}{\sqrt{y}}\right)\cdot(\sqrt{y})} = \lim\limits_{y\to 0+} \frac{\left(\frac{\sin(6y)}{6y}\right)\cdot 6\sqrt{y}}{\left(\frac{\sin(\sqrt{y})}{\sqrt{y}}\right)} = \frac{1\cdot 6\cdot 0}{1} = \boxed{0}$

7). $\lim\limits_{x\to 0} \frac{\sin(5x)}{\sin(3x)} = \lim\limits_{x\to 0} \frac{\frac{\sin(5x)}{5x}\cdot \cancel{5x}}{\frac{\sin(3x)}{3x}\cdot \cancel{3x}} = \lim\limits_{x\to 0} \frac{\left(\frac{\sin(5x)}{5x}\right)\cdot 5}{\left(\frac{\sin(3x)}{3x}\right)\cdot 3} = \frac{1\cdot 5}{1\cdot 3} = \boxed{\frac{5}{3}}$

$$\frac{d}{dx}(\sin x) = \cos x \quad ; \quad \frac{d}{dx}(\csc x) = -\csc(x)\cot(x)$$

$$\frac{d}{dx}(\cos x) = -\sin x \quad ; \quad \frac{d}{dx}(\sec x) = \sec(x)\tan(x)$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \quad ; \quad \frac{d}{dx}(\cot x) = -\csc^2(x)$$

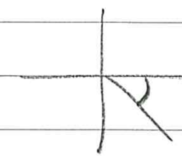
on formula Sheet

Examples:

1). $f(x) = 2\sin(x) + 5\cos(x)$

$f'(x) = 2\cos(x) - 5\sin(x)$; $\quad f'(-\pi/4) = 2 \cdot \frac{\sqrt{2}}{2} - 5\left(-\frac{\sqrt{2}}{2}\right)$

$f''(x) = -2\sin(x) - 5\cos(x)$; $\quad = \boxed{\frac{7\sqrt{2}}{2}}$



2). $f(x) = 3\sec(x) - 6$ Tangent line at $x = \pi/6$?

$f'(x) = 3\sec(x)\tan(x) \Rightarrow f'(\pi/6) = 3 \cdot \frac{1}{\sqrt{3}/2} \cdot \frac{1/2}{\sqrt{3}/2} = \underline{\underline{2}}$

$f(\pi/6) = 3 \cdot \frac{2}{\sqrt{3}} - 6 = 2\sqrt{3} - 6$

$y - (2\sqrt{3} - 6) = 2(x - \pi/6)$

$\boxed{y = 2(x - \pi/6) + (2\sqrt{3} - 6)}$

3). $f(x) = -3x\sin(x)\cos(x)$

$f'(x) = -3\sin(x)\cos(x) - 3x\cos^2(x) + 3x\sin^2(x) \Rightarrow f'(\pi) = \underline{-3\pi}.$

4). $p(t) = \frac{-4\tan(t) - 3}{\sec(t)}$; $\quad p'(t) = \frac{-4\sec^2(t)\sec(t) - (-4\tan t - 3)\sec t \tan t}{\sec^2(t)}$

5). $g(s) = 2s\tan(s)$; $\quad g'(s) = 2\tan(s) + 2s \cdot \sec^2(s).$

6). 77^{th} derivative of $\sin(x)$?

multiple of 4!

$f^{(0)}(x) = \sin(x)$ $\qquad f^{(4)}(x) = \sin(x)$ $\qquad f^{(76)}(x) = \sin(x)$

$f^{(1)}(x) = \cos(x)$ $\qquad f^{(5)}(x) = \cos(x)$ .... $\qquad f^{(77)}(x) = \boxed{\cos(x)}$

$f^{(2)}(x) = -\sin(x)$ $\qquad \vdots$

$f^{(3)}(x) = -\cos(x)$

2.4 Chain Rule

If g is differentiable at x, and f is differentiable at $g(x)$, then the composition $F = f \circ g$, given by $F(x) = f(g(x))$, is differentiable at x, and the derivative is given by:

$$\boxed{F'(x) = f'(g(x)) \cdot g'(x)} \qquad (f \circ g)' = f'(g) \cdot g'$$

In Leibniz notation: if $y = f(u)$ and $u = g(x)$ are both differentiable:

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

① $y = (x^4 + 13)^9$ $\Rightarrow$ $y'(x) = \underbrace{9(x^4+13)^8}_{\text{out}'(\text{in})} \cdot \underbrace{(4x^3)}_{(\text{in})'}$

power function = "outside function"

$x^4 + 13$ = "inside function"

General Principle: $(☺^n)' = n☺^{n-1} \cdot ☺'$

② $f(x) = \sqrt{3x^2 + 4x + 42}$ $\Rightarrow$ $f'(x) = \underbrace{\frac{1}{2\sqrt{3x^2+4x+42}}}_{\text{out}'(\text{in})} \cdot \underbrace{(6x+4)}_{(\text{in})'}$

Square root = "outside function"

$3x^2 + 4x + 42$ = "inside function"

$$(\sqrt{☺})' = \frac{1}{2\sqrt{☺}} \cdot ☺'$$

③ $f(x) = \underbrace{\cos}_{\text{out}}(\underbrace{x^3 + x^{7/2}}_{\text{in}})$ $\Rightarrow$ $f'(x) = \underbrace{-\sin(x^3 + x^{7/2})}_{\text{out}'(\text{in})} \cdot \underbrace{(3x^2 + \frac{7}{2}x^{5/2})}_{(\text{in})'}$

④ $f(x) = \sin^6(x) = (\sin x)^6$ $\Rightarrow$ $f'(x) = \underbrace{6(\sin x)^5}_{\text{out}'(\text{in})} \cdot \underbrace{(\cos x)}_{(\text{in})'}$

power function → outside

$\sin x$ → inside

⑤ $f(x) = \sin^6(3x^4) = (\sin(3x^4))^6$ Two Chain Rules here!

$$f'(x) = \underbrace{6(\sin(3x^4))^5}_{\text{out}'(\text{in})} \cdot \underbrace{(\underbrace{\cos(3x^4)}_{\text{out}'(\text{in})} \cdot \underbrace{12x^3}_{(\text{in})'})}_{(\text{in})'}$$

(6) $f(x) = 8\tan(\sin(4x))$

$$f'(x) = 8\sec^2(\sin(4x)) \cdot \overbrace{(4\cos(4x))}^{(\sin(4x))'}$$

(7) $f(x) = \sec(\sqrt{x})\tan\left(\frac{9}{x}\right)$ <u>PRODUCT RULE</u> first!

$$f'(x) = \underbrace{(\sec(\sqrt{x}))' \tan\left(\frac{9}{x}\right)} + \underbrace{\sec(\sqrt{x}) \cdot \left(\tan\left(\frac{9}{x}\right)\right)'}$$

$$\left(\left(\sec(\sqrt{x})\tan(\sqrt{x})\right) \cdot \frac{1}{2\sqrt{x}}\right)\tan\left(\frac{9}{x}\right) + \sec(\sqrt{x})\left(\sec^2\left(\frac{9}{x}\right) \cdot \frac{-9}{x^2}\right)$$

(8) $y = \sin\sqrt{\frac{4}{x+5}}$

$$y' = \left(\cos\sqrt{\frac{4}{x+5}}\right) \cdot \left(\sqrt{\frac{4}{x+5}}\right)' = \left(\cos\sqrt{\frac{4}{x+5}}\right) \cdot \frac{1}{2\sqrt{\frac{4}{x+5}}} \cdot \frac{-4}{(x+5)^2}$$

(9) $f(x) = (x^3 - 3x)^4$

$$f'(x) = 4(x^3 - 3x)^3 \cdot (3x^2 - 3)$$

$$f''(x) = \left(4 \cdot 3(x^3 - 3x)^2 \cdot (3x^2 - 3)\right) \cdot (3x^2 - 3) + 4(x^3 - 3x)^3 \cdot (6x)$$

$$= 12(x^3 - 3x)^2(3x^2 - 3)^2 + 24x(x^3 - 3x)^3$$

(10) $y = \sec^4(x^{2/3})$

$$y' = 4\sec^3(x^{2/3}) \cdot \sec(x^{2/3})\tan(x^{2/3}) \cdot \frac{2}{3}x^{-1/3}$$

$$= \frac{8}{3}x^{-1/3}\sec^4(x^{2/3})\tan(x^{2/3})$$

(11) $y = \sqrt{3x + \sqrt{2\sin(3x^2) + 4}}$

$$y' = \frac{1}{2\sqrt{3x + \sqrt{2\sin(3x^2) + 4}}}\left(3 + \frac{2\cos(3x^2) \cdot 6x}{2\sqrt{2\sin(3x^2) + 4}}\right)$$