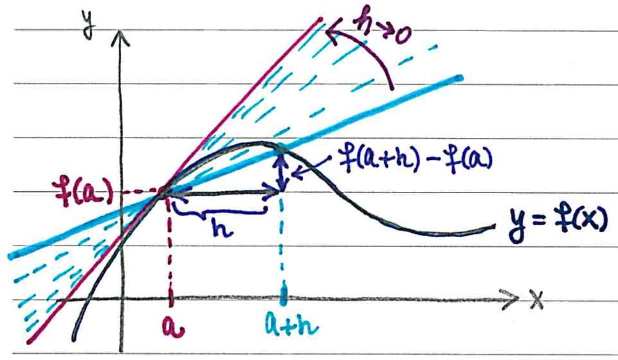
2.1. Derivatives & Rates of Change

Motivation 1: Tangent Lines



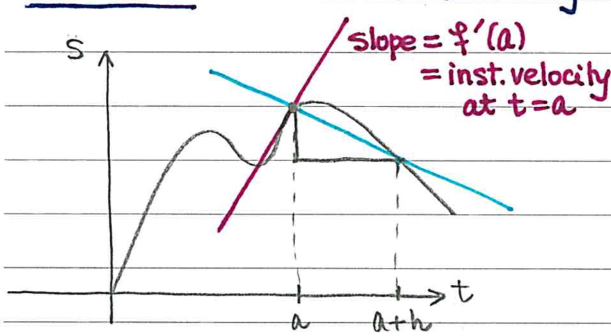
Find the slope of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$.

$$f'(a) := \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

(if the limit exists!)

Remember: h could be negative! (so approach from the left also in the picture).

Motivation 2: Instantaneous Velocity

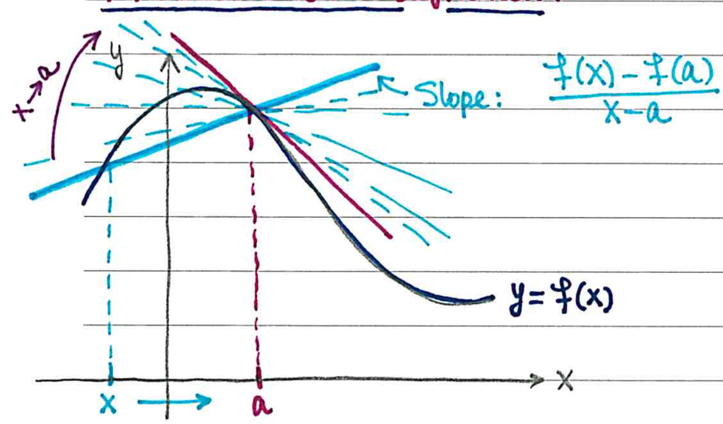


Position function $y = f(t)$
(tracks position of some particle at time t).
Average velocity over time interval $[a, a+h]$:

$$\frac{\Delta s}{\Delta t} = \frac{f(a+h) - f(a)}{(a+h) - (a)} = \frac{f(a+h) - f(a)}{h}$$

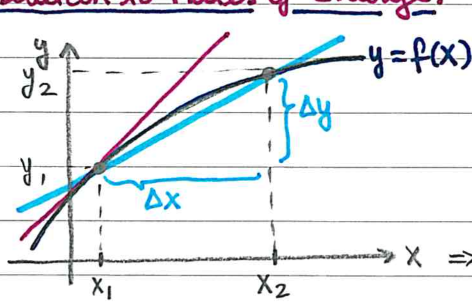
Letting $h \to 0$ means we are taking avg. velocity over smaller & smaller time intervals.

Alternative Limit Definition:



$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Relation to Rates of Change:



$\Delta x = x_2 - x_1$

$\Delta y = y_2 - y_1 = f(x_2) - f(x_1)$

Average rate of change of y with respect to x, on the interval $[x_1, x_2]$: $\boxed{\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}}$

$\Rightarrow$ Instantaneous r.o.c.: $\boxed{\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}}$

Example: $f(x) = 3\sqrt{x} - 2$ Find $f'(25)$ using the <u>limit definition</u>.

$$f'(25) = \lim_{h\to 0} \frac{f(25+h) - f(25)}{h}$$

$$= \lim_{h\to 0} \frac{(3\sqrt{25+h} - 2) - (3\sqrt{25} - 2)}{h}$$

$$= \lim_{h\to 0} \frac{3(\sqrt{25+h} - \sqrt{25})}{h} \cdot \frac{\sqrt{25+h} + \sqrt{25}}{\sqrt{25+h} + \sqrt{25}}$$

$$= \lim_{h\to 0} \frac{3((25+h) - 25)}{h(\sqrt{25+h} + \sqrt{25})}$$

$$= \lim_{h\to 0} \frac{3\cancel{h}}{\cancel{h}(\sqrt{25+h} + 5)}$$

$$= \lim_{h\to 0} \frac{3}{\sqrt{25+h} + 5} = \frac{3}{5+5} = \boxed{\frac{3}{10}} \Rightarrow \boxed{f'(25) = \frac{3}{10}}$$

Equation of line tangent to $f(x) = y$ at $(25, \underbrace{f(25)}_{13})$?

Point: $(25, 12)$

Slope: $3/10$

$$y - 13 = \frac{3}{10}(x - 25) \qquad \boxed{y = \frac{3}{10}(x-25) + 13}$$

Example: $f(x) = \frac{1}{x+3}$ Find $f'(-2)$ using the <u>limit definition</u>.

$$f'(-2) = \lim_{h\to 0} \frac{f(-2+h) - f(-2)}{h} = \lim_{h\to 0} \frac{\frac{1}{(-2+h)+3} - 1}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{h+1} - 1}{h} = \lim_{h\to 0} \frac{\cancel{1} - h - \cancel{1}}{h(h+1)} = \lim_{h\to 0} \frac{-1}{h+1} = \boxed{-1} \Rightarrow \boxed{f'(-2) = -1}$$

Equation of line tangent to $y = f(x)$ at $(-2, \underbrace{f(-2)}_{=1})$?

Point: $(-2, 1)$

Slope: -1

$$y - 1 = -1(x + 2)$$

$$y = -x - 2 + 1 \qquad \boxed{y = -x - 1}$$

2.2 The Derivative as a Function

Let a vary in the previous definition, and define:

$$\boxed{f'(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}} \quad (*) \leftarrow \text{The derivative of } f$$

(wherever the limit exists).

* We say f is **differentiable at a** if $f'(a)$ exists (the limit in $(*)$ exists).
* We say f is **differentiable** on some interval (a,b) (or (a,∞), $(-\infty,a)$, $(-\infty,\infty)$), if it is differentiable at every number in the interval.

Other Notations: For the derivative of $y = f(x)$, we may write (either one):

$$f'(x) \; ; \; y' \; ; \; \frac{dy}{dx} \; ; \; \frac{df}{dx} \; ; \; \frac{d}{dx} f(x)$$

And for a specific value of $f(x)$ at some point $x = a$: $f'(a)$; $\left.\frac{dy}{dx}\right|_{x=a}$; $\left.\frac{df}{dx}\right|_{x=a}$.

Higher Derivatives: $(f')' = f''$ (the second derivative of f)

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} \quad \text{(Leibniz notation)}$$

$$\left.\begin{aligned} (f'')' &= f''' \\ \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) &= \frac{d^3y}{dx^3} \end{aligned}\right\} \text{Third derivative} \quad \ldots \quad \boxed{f^{(n)}(x) = \frac{d^ny}{dx^n} = y^{(n)}}$$

n^{th} derivative.

Example: $f(x) = 3\sqrt{x} - 2$ Find the general formula for $f'(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(3\sqrt{x+h} - \cancel{2}) - (3\sqrt{x} - \cancel{2})}{h}$$

$$= \lim_{h \to 0} \frac{3(\sqrt{x+h} - \sqrt{x})}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{3(\cancel{x}+h-\cancel{x})}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{3h}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{3}{\sqrt{x+h} + \sqrt{x}} = \boxed{\frac{3}{2\sqrt{x}}} \Rightarrow \boxed{f'(x) = \frac{3}{2\sqrt{x}}}$$

So, for example: $f'(25) = \frac{3}{2\sqrt{25}} = \frac{3}{10}$ ✓

$f'(2) = \frac{3}{2\sqrt{2}}$, $f'(9) = \frac{1}{2}$ etc

Example: $f(x) = x^2 - 2x + 1$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\left((x+h)^2 - 2(x+h) + 1\right) - (x^2 - 2x + 1)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{2x} - 2h + \cancel{1} - \cancel{x^2} + \cancel{2x} - \cancel{1}}{h}$$

$$= \lim_{h \to 0} \frac{h^2 + 2xh - 2h}{h}$$

$$= \lim_{h \to 0} (h + 2x - 2) = \boxed{2x - 2} \Rightarrow \boxed{f'(x) = 2x - 2}$$

DIFFERENTIABLE $\Rightarrow$ CONTINUOUS

If a function f is differentiable at $x=a$, then it is continuous at $x=a$.

f diff'ble at $x=a$ means that the limit $f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ exists. (and is finite)

$$\Rightarrow \lim_{x\to a}\left[f(x)-f(a)\right]=\lim_{x\to a}\left(\frac{f(x)-f(a)}{(x-a)}(x-a)\right)=f'(a)\cdot 0=\boxed{0}$$

$$\Rightarrow \lim_{x\to a}f(x)=f(a) \Rightarrow f \text{ is continuous at } x=a.$$

CONTINUOUS $\not\Rightarrow$ DIFFERENTIABLE

A function f could be continuous at $x=a$ but not differentiable there.

Example 1: $f(x)=\begin{cases}-7x^2+5x, & \text{if } x\le 0\\ 3x^2+5x, & \text{if } x>0\end{cases}$

$\lim_{x\to 0^-}f(x)=\lim_{x\to 0+}f(x)=0=f(0) \Rightarrow$ continuous at $x=0$

$$\left.\begin{aligned}&\lim_{h\to 0_-}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0_-}\frac{-7h^2+5h}{h}=\lim_{h\to 0_-}(-7h+5)=\boxed{5}\\ &\lim_{h\to 0_+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0_+}\frac{3h^2+5h}{h}=\lim_{h\to 0_+}(3h+5)=\boxed{5}\end{aligned}\right\}\text{ they agree}$$

$\Rightarrow \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\boxed{5=f'(0)} \Rightarrow f$ is also differentiable at $x=0$.

Example 2: $f(x)=\begin{cases}-2x^2+3x, & \text{if } x\le 0\\ 4x^2-3x, & \text{if } x>0\end{cases}$

$\lim_{x\to 0-}f(x)=\lim_{x\to 0+}f(x)=0=f(0) \Rightarrow f$ is continuous at $x=0$.

$$\left.\begin{aligned}&\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^-}(-2h+3)=\boxed{3}\\ &\lim_{h\to 0+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0+}(4h-3)=\boxed{-3}\end{aligned}\right\}\text{ they do NOT agree!}$$

$\Rightarrow f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$ DNE $\Rightarrow f$ is **NOT** differentiable at $x=0$