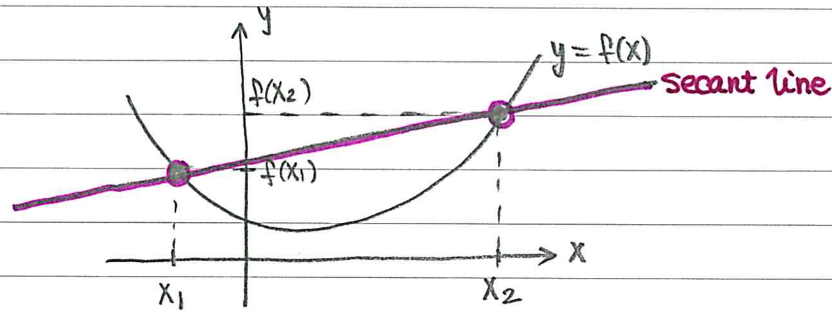
# Chapter 1 - Summary

## 1.4. * Secant Lines: *

| | | | | |
|---|---|---|---|---|
| The average rate of change of of a function $f(x)$ over the interval $[x_1, x_2]$ | = | Slope of the secant line of $f(x)$ determined by $(x_1, f(x_1))$ & $(x_2, f(x_2))$ | = | $\dfrac{f(x_2)-f(x_1)}{x_2-x_1}$ |




**Example:** Find the slope of the secant line of $f(x) = -2\tan(x)+5$ over $[0, \pi/6]$

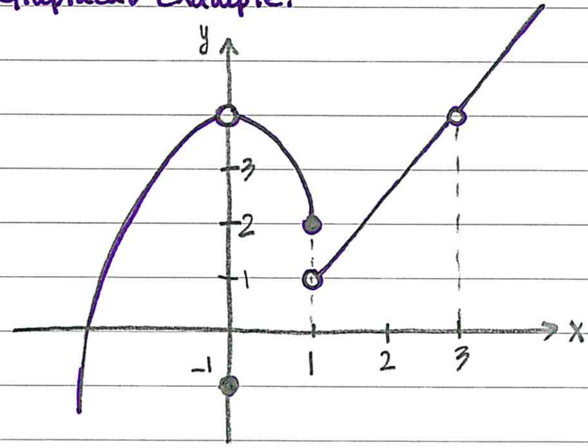
$$\frac{f(\pi/6)-f(0)}{\pi/6 - 0} = \frac{(-2\tan(\pi/6)+5)-(-2\tan(0)+5)}{\pi/6} = \frac{-2\cdot 1/\sqrt{3}}{\pi/6} = \frac{-2}{\sqrt{3}}\cdot\frac{6}{\pi}$$

$$= \frac{-4\sqrt{3}}{\pi}$$

## 1.5. * The Limit of a Function: *

**Def:** Suppose $f(x)$ is defined on an open interval containing $x=a$ (except possibly at $x=a$ itself). We say that $\lim_{x\to a} f(x) = L$ if we can make the values of $f(x)$ arbitrarily close to $L$ by taking $x$ sufficiently close to $a$ (but not equal to

Graphical Example:




$\lim_{x\to 0} f(x) = 4$ ; $f(0) = -1$

$\underbrace{\lim_{x\to 1^-} f(x) = 2 \text{ ; } \lim_{x\to 1^+} f(x) = 1}_{\lim_{x\to 1} f(x) \text{ DNE}}$ ; $f(1) = 2$

$\lim_{x\to 3} f(x) = 4$ ; $f(3) = $ DNE.

## One-Sided Limits:

$\boxed{\lim_{x\to a^-} f(x) = L}$ → If we can make $f(x)$ arbitrarily close to $L$ by taking $x$ sufficiently close to $a$ and less than $a$.

$\boxed{\lim_{x\to a^+} f(x) = L}$ → If we can make $f(x)$ arbitrarily close to $L$ by taking $x$ sufficiently close to $a$ and larger than $a$.

$$\boxed{\lim_{x\to a} f(x) = L} \iff \boxed{\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L}$$

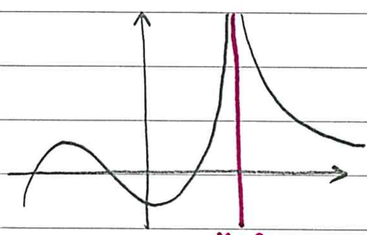
## Infinite Limits:

$$\frac{1}{0_-} = -\infty \quad \& \quad \frac{1}{0_+} = +\infty$$

**Vertical Asymptote:** The line $x=a$ is called a vertical asymptote of $y=f(x)$ if at least one of the following holds:

$\lim_{x\to a} f(x) = +\infty$ ; $\lim_{x\to a^-} f(x) = +\infty$ ; $\lim_{x\to a^+} f(x) = +\infty$ ;

$\lim_{x\to a} f(x) = -\infty$ ; $\lim_{x\to a^+} f(x) = -\infty$ ; $\lim_{x\to a^+} f(x) = -\infty$ .

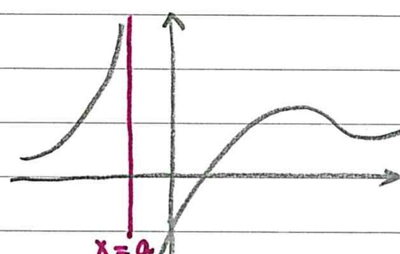


$x=a$

$\lim_{x\to a^-} f(x) = +\infty$

$\lim_{x\to a^+} f(x) = +\infty$

$\lim_{x\to a} f(x) = +\infty$



$x=a$

$\lim_{x\to a^-} f(x) = +\infty$

$\lim_{x\to a^+} f(x) = -\infty$

$\lim_{x\to a} f(x)$ DNE

**Examples**: Find the vertical asymptotes:

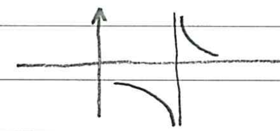
1). $f(x) = \dfrac{2x-2}{(x-3)(x+1)} = \dfrac{2(x-1)}{(x-3)(x+1)} \Rightarrow \boxed{\text{V.A.: } x=3,\ x=-1}$

2). $f(x) = \dfrac{2x+2}{(x-3)(x+1)} = \dfrac{2\cancel{(x+1)}}{(x-3)\cancel{(x+1)}} = \dfrac{2}{x-3} \Rightarrow \boxed{\text{V.A.: } x=3}$

**!** Always check for common factors !!

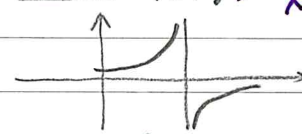
3). $f(x) = \dfrac{x}{2x^2+5}$ No Vertical Asymptotes ($2x^2+5=0$ has no real roots)

4). $f(x) = \dfrac{8}{x-5}$ $\boxed{\text{V.A.: } x=5}$



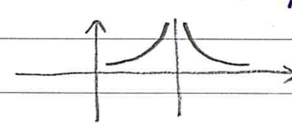
$\lim\limits_{x\to 5^-} f(x) = \boxed{-\infty}$ $\left(\dfrac{1}{0-}\right)$; $\lim\limits_{x\to 5^+} f(x) = \boxed{+\infty}$ $\left(\dfrac{1}{0+}\right)$; $\lim\limits_{x\to 5} f(x) = \text{DNE}$

5). $f(x) = \dfrac{8}{5-x}$ $\boxed{\text{V.A. : } x=5}$



$\lim\limits_{x\to 5^-} f(x) = \boxed{+\infty}$ $\left(\dfrac{1}{0+}\right)$; $\lim\limits_{x\to 5^+} f(x) = \boxed{-\infty}$ $\left(\dfrac{1}{0-}\right)$; $\lim\limits_{x\to 5} f(x) = \text{DNE}$

6). $f(x) = \dfrac{8}{|x-5|}$ $\boxed{\text{V.A.: } x=5}$



$\lim\limits_{x\to 5^-} f(x) = \boxed{+\infty}$ $\left(\dfrac{1}{0+}\right)$; $\lim\limits_{x\to 5^+} f(x) = \boxed{+\infty}$ $\left(\dfrac{1}{0+}\right)$; $\lim\limits_{x\to 5} f(x) = +\infty$

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## 1.6. * Limit Laws *

$$\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$$

$$\lim_{x\to a}[c\cdot f(x)] = c\cdot \lim_{x\to a} f(x)$$

$$\lim_{x\to a}[f(x)\cdot g(x)] = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x)$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} \quad \text{if } \lim_{x\to a} g(x) \neq 0$$

if both $\lim\limits_{x\to a} f(x)$ & $\lim\limits_{x\to a} g(x)$ EXIST and are FINITE.

## $\boxed{\frac{0}{0}}$-type limits : FACTOR!

① $\lim\limits_{x\to 3} \dfrac{x^2-9}{x-3} = \lim\limits_{x\to 3} \dfrac{\cancel{(x-3)}(x+3)}{\cancel{x-3}} = \lim\limits_{x\to 3}(x+3) = \boxed{6}$

② $\lim\limits_{x\to 1} \dfrac{x-1}{x^2+8x-9} = \lim\limits_{x\to 1} \dfrac{\cancel{x-1}}{\cancel{(x-1)}(x+9)} = \lim\limits_{x\to 1} \dfrac{1}{x+9} = \boxed{\frac{1}{10}}$

③ $\lim\limits_{x\to 2} \dfrac{x^2+5x-14}{x^2-2x} = \lim\limits_{x\to 2} \dfrac{\cancel{(x-2)}(x+7)}{x\cancel{(x-2)}} = \lim\limits_{x\to 2} \dfrac{x+7}{x} = \boxed{\frac{9}{2}}$

④ $\lim\limits_{x\to 49} \dfrac{\sqrt{x}-7}{x-49} = \lim\limits_{x\to 49} \dfrac{\cancel{\sqrt{x}-7}}{\cancel{(\sqrt{x}-7)}(\sqrt{x}+7)} = \lim\limits_{x\to 49} \dfrac{1}{\sqrt{x}+7} = \boxed{\frac{1}{14}}$

⑤ $\lim\limits_{x\to 6^-} \dfrac{|x-6|}{x-6} = \lim\limits_{x\to 6^-} \dfrac{-\cancel{(x-6)}}{\cancel{x-6}} = \boxed{-1}$

$\lim\limits_{x\to 6^+} \dfrac{|x-6|}{x-6} = \lim\limits_{x\to 6^+} \dfrac{+\cancel{(x-6)}}{\cancel{x-6}} = \boxed{+1}$

$\Rightarrow \lim\limits_{x\to 6} \dfrac{|x-6|}{x-6}$ dne

$$\boxed{|☺| = \begin{cases} ☺ & \text{if } ☺ \geq 0 \\ -☺ & \text{if } ☺ < 0 \end{cases}}$$

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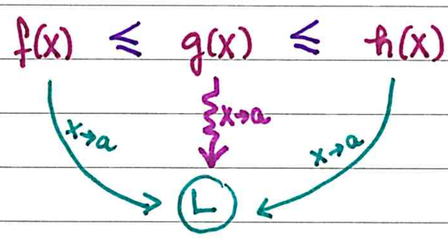
## Squeeze Theorem:

If $f(x) \leq g(x) \leq h(x)$ when $x$ is near $a$, and $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} h(x) = L$, then $\lim\limits_{x\to a} g(x) = L$.




**Example:** Suppose $7\cos(5x) \leq g(x) \leq 7+x^2$ for all real $x$.
Then, since

$$\lim_{x\to 0}\left(7\cos(5x)\right) = \lim_{x\to 0}\left(7+x^2\right) = 7$$

by the Squeeze Theorem:

$$\lim_{x\to 0} g(x) = 7.$$

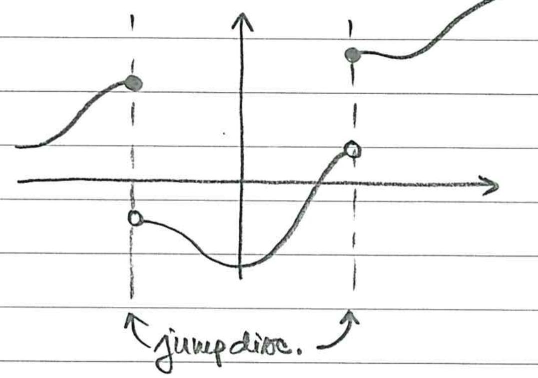
## 1.8. * Continuity *

**Def.**: A function $f$ is continuous at $x=a$ if $\boxed{\lim_{x\to a} f(x) = f(a)}$.

**Types of Discontinuities at a point $x=a$:**

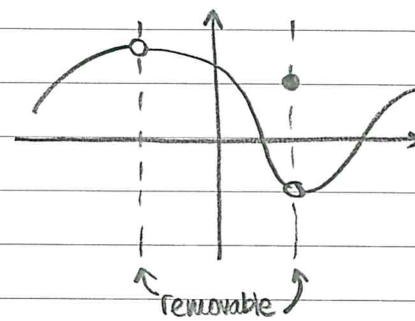
1. **Jump:** $\boxed{\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)}$

~ the side limits exist and are finite, but they do not agree. ~

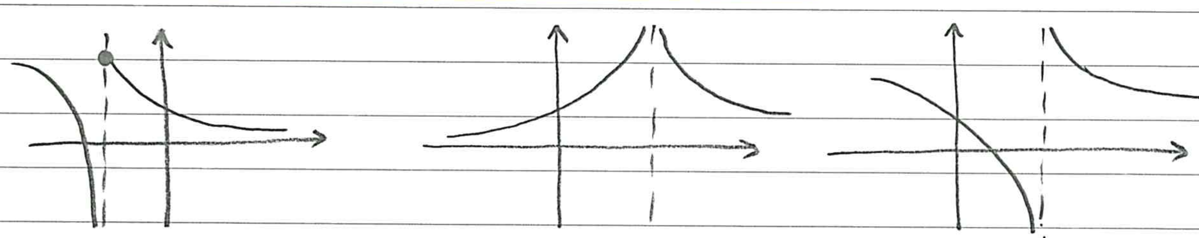



2. **Removable:** $\boxed{\lim_{x\to a} f(x) \neq f(a)}$

~ the limit of $f(x)$ as $x\to a$ exists, but it differs from $f(a)$ – or $f(a)$ does not exist! ~




3. **Infinite:** $\boxed{\text{At least one of } \lim_{x\to a^\pm} f(x) \text{ is } \pm\infty.}$



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**Examples:**

1). $f(x) = \dfrac{x^2-9}{x^2-7x+12} = \dfrac{\cancel{(x-3)}(x+3)}{\cancel{(x-3)}(x-4)} = \dfrac{x+3}{x-4}$ (as long as $x\neq 3$)

Remark: As given, $f$ is not defined at $x=3$, but $\lim_{x\to 3} f(x) = \lim_{x\to 3} \dfrac{x+3}{x-4} = -6$, so $f$ has a removable discontinuity at $x=3$.

Interval of continuity for $f$: $(-\infty,3)\cup(3,4)\cup(4,\infty)$

removable disc. @ $x=3$    infinite disc. @ $x=4$

Vertical Asymptote of $f$: $x=4$.

2). $f(x) = \sqrt{6x-8}$   $f$ is continuous on its domain $[4/3, \infty)$.

$6x - 8 \geq 0$ ; $6x \geq 8$ ; $x \geq 4/3$

3). $f(x) = \dfrac{x}{\sqrt{6x-8}}$   $f$ is continuous on its domain $(4/3, \infty)$

(has infinite discontinuity at $x = 4/3$). $\lim\limits_{x \to 4/3^+} f(x) = +\infty \left(\frac{1}{0+}\right)$

4). $f(x) = \begin{cases} x^2+6x+5, & \text{if } x<3 \\ 5, & \text{if } x=3 \\ -2x+4, & \text{if } x>3 \end{cases}$

$\lim\limits_{x \to 3^-} f(x) = (3)^2 + 6(3) + 5 = 32$

$\lim\limits_{x \to 3^+} f(x) = -2(3) + 4 = -2$

$\Rightarrow$ Jump discontinuity at $x=3$.

5). $f(x) = \begin{cases} \dfrac{x^2-9}{x-3}, & \text{if } x \neq 3 \\ 10, & \text{if } x=3 \end{cases}$

$\lim\limits_{x \to 3} f(x) = \lim\limits_{x \to 3} (x+3) = \boxed{6} \neq f(3)$

$\Rightarrow$ Removable discontinuity at $x=3$

6). $f(x) = \begin{cases} x^2-8, & \text{if } x \leq c \\ 4x-12, & \text{if } x>c \end{cases}$   For what $c$ is $f$ continuous?

$\left.\begin{array}{l} \lim\limits_{x \to c^-} f(x) = c^2 - 8 = f(c) \\ \lim\limits_{x \to c^+} f(x) = 4c - 12 \end{array}\right\} \Rightarrow c^2 - 8 = 4c - 12$

$c^2 - 4c + 4 = 0$

$(c-2)^2 = 0$   $\boxed{c=2}$

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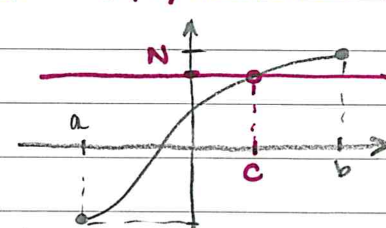
**Intermediate Value Theorem:** Let $f$ be continuous on $[a,b]$ and $N$ be any number between $f(a)$ and $f(b)$. Then there is a value $c \in (a,b)$ such that $f(c) = N$.

**Example:** Show that $\dfrac{1}{x-2} + \dfrac{1}{x-9} = 0$ has at least one solution in the interval $(2,9)$.




$f(x) = \dfrac{1}{x-2} + \dfrac{1}{x-9}$ is continuous on $(-\infty, 2) \cup (2,9) \cup (9, \infty)$.

$\left.\begin{array}{l} f(3) = \frac{1}{1} - \frac{1}{6} = \frac{5}{6} > 0 \\ f(8) = \frac{1}{6} - 1 = -\frac{5}{6} < 0 \end{array}\right\} \Rightarrow$ by IVT, $f(x)=0$ has a solution on the interval $(3,8)$ and therefore also on $(2,9)$.