# Some Non-Linear Equations (Clever Substitutions).

$$(x-2x^2y)\frac{dy}{dx}+y+2xy^2=0 \qquad x(1-2xy)\frac{dy}{dx}+y(1+2xy)=0$$

Substitution: $u=2xy \Rightarrow y=\frac{u}{2x}$

$$\Rightarrow \frac{dy}{dx}=\frac{2x\frac{du}{dx}-2u}{4x^2}=\frac{1}{2x}\frac{du}{dx}-\frac{1}{2x^2}u$$

$$x(1-u)\left(\frac{1}{2x}\frac{du}{dx}-\frac{1}{2x^2}u\right)+\frac{u}{2x}(1+u)=0$$

$$\frac{1}{2}(1-u)\frac{du}{dx}-\frac{1}{2x}(u-u^2)+\frac{1}{2x}(u+u^2)=0$$

$$\frac{1}{2}(1-u)\frac{du}{dx}+\frac{1}{x}u^2=0 \Rightarrow \underbrace{\frac{u^2}{x}=\frac{u-1}{2}\frac{du}{dx}}_{\text{Separable!}} \Rightarrow \boxed{\frac{2}{x}dx=\frac{u-1}{u^2}du}$$

Unless u=0, in which case 2xy=0, so y=0. Solution? Yes: y=0

$$\Rightarrow 2\ln|x|=\ln|u|+\frac{1}{u}+C \Rightarrow \ln\left|\frac{x^2}{2xy}\right|=\frac{1}{2xy}+C$$

$$\Rightarrow \ln\left|\frac{x}{2y}\right|=\frac{1}{2xy}+C \Rightarrow \left|\frac{x}{2y}\right|=Ce^{\frac{1}{2xy}}$$

$$\Rightarrow \boxed{x=Cye^{\frac{1}{2xy}}}$$

$$2xy\frac{dy}{dx}+2y^2=3x-6$$

Substitution: $u=y^2 \Rightarrow \frac{du}{dx}=2y\frac{dy}{dx}$

$$\boxed{x\frac{du}{dx}+2u=3x-6} \leftarrow \text{Linear!}$$

$$\frac{du}{dx}+\frac{2}{x}u=3-\frac{6}{x}$$

$$\mu(x)=e^{\int(2/x)dx}=e^{2\ln|x|}=x^2 \Rightarrow \frac{d}{dx}(x^2u)=3x^2-6x \Rightarrow x^2u=x^3-3x^2+C$$

$$\Rightarrow \boxed{x^2y^2=x^3-3x^2+C}$$

$$x\frac{dy}{dx}-y=\frac{x^3}{y}e^{y/x}$$

Substitution: $u=\frac{y}{x} \Rightarrow y=ux \Rightarrow \frac{dy}{dx}=x\frac{du}{dx}+u$

$$\Rightarrow x^2\frac{du}{dx}+\cancel{xu}-\cancel{ux}=\frac{x^3}{ux}e^u \Rightarrow \boxed{\frac{du}{dx}=\frac{1}{u}e^u} \Rightarrow ue^{-u}du=dx$$

Separable!

$$\Rightarrow -ue^{-u}-e^{-u}=x+c \Rightarrow -e^{-u}(u+1)=x+c \Rightarrow u+1=(c-x)e^u$$

$$\Rightarrow \frac{y}{x}+1=(c-x)e^{y/x} \Rightarrow \boxed{y+x=(c-x)xe^{y/x}}$$

$$y''=2x(y')^2$$

Substitution: $u=y' \Rightarrow \frac{du}{dx}=y'' \Rightarrow \frac{du}{dx}=2xu^2 \Rightarrow \frac{du}{u^2}=2x\,dx$

Unless u=0,
in which case
y'=0, so y=c.
Solution? Yes:
y=c
(all constant
functions
are solutions as
well)

$$\Rightarrow -\frac{1}{u}=x^2+c \Rightarrow -\frac{1}{y'}=x^2+c \Rightarrow \boxed{y'=-\frac{1}{x^2+c}}$$

Case 1: $\boxed{c=0} \Rightarrow y'=-\frac{1}{x^2} \Rightarrow \boxed{y_1=\frac{1}{x}+c_1}$

Case 2: $\boxed{c=a^2>0} \Rightarrow y'=-\frac{1}{x^2+a^2} \Rightarrow \boxed{y_2=-\frac{1}{a}\arctan\left(\frac{x}{a}\right)+c_2}$

Case 3: $\boxed{c=-a^2<0} \Rightarrow y'=-\frac{1}{x^2-a^2}=-\frac{1}{(x-a)(x+a)}=-\frac{1}{2a}\frac{(x+a)-(x-a)}{(x-a)(x+a)}$

$$\Rightarrow y'=-\frac{1}{2a}\left(\frac{1}{x-a}-\frac{1}{x+a}\right)$$

$$\Rightarrow y=\frac{1}{2a}\left(\ln|x+a|-\ln|x-a|\right)+c_3$$

$$\Rightarrow \boxed{y_3=\frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right|+c_3}$$

$$xe^{2y}\frac{dy}{dx}+e^{2y}=\frac{\ln x}{x}$$

$$u=e^{2y} \Rightarrow \frac{du}{dx}=2e^{2y}\cdot\frac{dy}{dx}$$

$$\Rightarrow x\cdot\frac{1}{2}\frac{du}{dx}+u=\frac{\ln x}{x} \Rightarrow \frac{du}{dx}+\frac{2}{x}u=\frac{2\ln x}{x^2}$$

(Linear!)

$$\mu=x^2 \Rightarrow \frac{d}{dx}(x^2u)=2\ln x$$

$$\Rightarrow x^2u=\int 2\ln x\,dx=2x\ln x-2x+C \Rightarrow \boxed{x^2e^{2y}=2x\ln x-2x+C}$$

$$x^4y^2y'+x^3y^3=2x^3-3$$

$$u=y^3 \Rightarrow \frac{du}{dx}=3y^2\frac{dy}{dx} \Rightarrow x^4\cdot\frac{1}{3}\frac{du}{dx}+x^3u=2x^3-3$$ (linear!)

$$\frac{du}{dx}+\frac{3}{x}u=\frac{6}{x}-\frac{9}{x^4}$$

$$\mu(x)=x^3$$

$$\Rightarrow \frac{d}{dx}(x^3u)=6x^2-\frac{9}{x}$$

$$\Rightarrow \boxed{x^3y^3=2x^3-9\ln|x|+C}$$

⊛ $2yy'+x^2+y^2+x=0.$

$u=y^2 \Rightarrow \frac{du}{dx}=2yy' \Rightarrow \frac{du}{dx}+x^2+u+x=0$

$\Rightarrow \frac{du}{dx}+u=-x^2-x$ (linear)

$\mu=e^x \Rightarrow \frac{d}{dx}(ue^x)=(-x^2-x)e^x$

$\int xe^x dx = xe^x - e^x$

$\int x^2e^x dx = x^2e^x - 2\int xe^x dx$

$= x^2e^x - 2xe^x + 2e^x$

$\Rightarrow ue^x = (-x^2+2x-2-x+1)e^x + c$

$\Rightarrow y^2e^x = (-x^2+x-1)e^x + c$

$\Rightarrow \boxed{y^2=-x^2+x-1+ce^{-x}}$

⊛ $(1+ye^x)y'+y=0$

$u=ye^x \Rightarrow \frac{du}{dx}=y'e^x+ye^x=y'e^x+u \Rightarrow y'e^x=\frac{du}{dx}-u$

$(1+u)\frac{1}{e^x}(u'-u)+\frac{1}{e^x}u=0$

$(1+u)(u'-u)+u=0$

$u'-\cancel{u}+uu'-u^2+\cancel{u}=0$

$(1+u)u'-u^2=0$ $\quad (u=-1 \Rightarrow 1=0$ ↯ $\Rightarrow u\neq -1)$

$u'=\frac{u^2}{1+u}$ Autonomous

$\frac{du}{dx}=\frac{u^2}{1+u} \Rightarrow \frac{1+u}{u^2}du=dx \Rightarrow x=-\frac{1}{u}+\ln|u|+c$

$\Rightarrow x=-\frac{1}{ye^x}+\ln|ye^x|+c$

$\Rightarrow \cancel{x}=-\frac{1}{ye^x}+\ln|y|+\cancel{x}+c \Rightarrow \boxed{\ln|y|=\frac{1}{ye^x}+c}$

$\Rightarrow \boxed{y=ce^{\frac{1}{ye^x}}}$ or $y\ln|y|=e^{-x}+cy \Rightarrow \boxed{e^{-x}=y\ln|y|+cy}$