Higher Order Linear ODEs

(A) n^{th} Order Linear Homogeneous Equations

$$\boxed{a_n(x)y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)+\ldots+a_1(x)y'(x)+a_0(x)y(x)=0} \quad (*)$$

(assume throughout that we are working on an interval I where the Existence & Uniqueness Thm. hold)

Superposition Principle:

If $y_1, y_2, \ldots, y_K$ are solutions to $(*)$, where K is some positive integer, then any linear combination:

$$y=c_1y_1+c_2y_2+\ldots+c_Ky_K$$

is also a solution to $(*)$.

To see this, look at the case $K=2$, so suppose y_1 and y_2 are solutions to $(*)$:

$$\begin{cases} a_n y_1^{(n)}+a_{n-1}y_1^{(n-1)}+\ldots+a_1y_1'+a_0y_1=0 & \mid \cdot c_1 \quad \text{(some constant)} \\ a_n y_2^{(n)}+a_{n-1}y_2^{(n-1)}+\ldots+a_1y_2'+a_0y_2=0 & \mid \cdot c_2 \quad \text{(some constant)} \end{cases}$$

$$\Rightarrow \begin{cases} a_n(c_1y_1)^{(n)}+a_{n-1}(c_1y_1)^{(n-1)}+\ldots+a_1(c_1y_1)'+a_0(c_1y_1)=0 \\ a_n(c_2y_2)^{(n)}+a_{n-1}(c_2y_2)^{(n-1)}+\ldots+a_1(c_2y_2)'+a_0(c_2y_2)=0 \end{cases}$$

$\oplus$

$$a_n(c_1y_1+c_2y_2)^{(n)}+\ldots+a_1(c_1y_1+c_2y_2)'+a_0(c_1y_1+c_2y_2)=0$$

$\Rightarrow c_1y_1+c_2y_2$ is also a solution!

Take a moment to remark why this can only work for homogeneous ODEs!

Some Useful Consequences:

1. If y_1 is a solution to $(*)$, then any constant multiple $y=cy_1$ is also a solution. (Take $K=1$ above)

2. The trivial solution $y=0$ satisfies any homogeneous linear ODE. (Take $c_1=\ldots=c_K=0$ or just verify).

Fundamental Sets

Def.: A fundamental set of solutions to an n^{th} order linear homogeneous ODE (*) on some interval I is a set:

$$\{y_1, y_2, \ldots, y_n\}$$

of n solutions that are linearly independent on I.

- Two very important facts about fundamental sets:
  1). If we have a fundamental set, we have all the solutions.
  2). An "if and only if" relationship to the Wronskian.

1 Let $\{y_1, y_2, \ldots, y_n\}$ be a fundamental set of solutions to (*) on some interval I. Then ANY solution to (*) on I is a linear combination of $y_1, \ldots, y_n$.

For this reason, if $\{y_1, \ldots, y_n\}$ is a fundamental set, then

$$\boxed{y = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n}$$

is called the general solution to (*) on I.

This is actually a simple consequence of the Existence & Uniqueness Theorem and a result from linear algebra, which we recall below:

Linear Algebra Flashback:

Recall that a system of n linear algebraic equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \end{cases}$$

can be expressed in matrix form as $A\vec{x} = \vec{b}$, where A is the $n \times n$ matrix of coefficients, $\vec{x}$ is the vector of unknowns.

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}; \quad \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}; \quad \vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}.$$

Possible behaviors of such a system, in terms of the determinant det(A):

- **det(A) ≠ 0** ⇒ unique solution
- **det(A) = 0** ⇒ either no solution or infinitely many solutions.

  [Important special case: if the system is homogeneous, i.e. $\vec{b} = \vec{0}$ and $\det(A) = 0$, then there are infinitely many solutions]

Back to our homogeneous ODEs, look at the case $\boxed{n=2}$.

Equation:

$$\boxed{a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0} \qquad (*)$$

(assume a_2, a_1, a_0 are continuous on I and $a_2(x) \neq 0, \forall x \in I \rightsquigarrow \exists!$)

Suppose:
- $Y(x)$ = a solution to $(*)$
- $\{y_1(x), y_2(x)\}$ = a <u>fundamental set</u> for $(*)$ (i.e. <u>lin. indp.</u> solutions).

Why should Y be a linear combination of y_1 and y_2?

- Let $x_0 \in I$ be a point where the Wronskian $W(y_1, y_2)(x_0) \neq 0$ (we know such a point must exist, since y_1, y_2 are <u>linearly independent</u>).
- Let $b_1 = Y(x_0)$, $b_2 = Y(x_0)$ and look at the system:

$$\begin{cases} y_1(x_0)C_1 + y_2(x_0)C_2 = b_1 \\ y_1'(x_0)C_1 + y_2'(x_0)C_2 = b_2 \end{cases}$$

(C_1, C_2 = the unknowns; $y_{1,2}(x_0), y_{1,2}'(x_0)$ = the coefficients)

The determinant of coefficients:

$$\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = W(y_1, y_2)(x_0) \neq 0$$

$\Rightarrow$ the system has a unique solution C_1, C_2.

- Then look at the function $Z(x) = C_1 y_1(x) + C_2 y_2(x)$:
  - $*$ Z is a solution to $(*)$ $\rightsquigarrow$ <u>Superposition</u>
  - $*$ $Z(x_0) = b_1$ and $Z(x_0) = b_2$

$\Rightarrow$ Y and Z are <u>both</u> solutions on I to the <u>IVP</u> $\begin{cases} a_2 y'' + a_1 y' + a_0 y = 0 \\ y(x_0) = b_1;\ y'(x_0) = b_2 \end{cases}$

$\Rightarrow$ $\boxed{Y = Z}$ by the Existence & Uniqueness Theorem

$\Rightarrow$ $Y = C_1 y_1 + C_2 y_2$. ■

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(2) A set $\{y_1, \dots, y_n\}$ of solutions to $(*)$ is a fundamental set (i.e. they are linearly independent) on an interval I <u>if and only if</u> the Wronskian:

$$\boxed{W(y_1, y_2, \dots, y_n)(x) \neq 0, \ \forall x \in I.}$$

$*$ <u>What is special here?</u> One implication is easy: we know from last time that if $W(f_1, \dots, f_n)(x) \neq 0$ for <u>at least one</u> $x \in I$, then $\{f_1, \dots, f_n\}$ are lin. indep. (so obviously also if $W \neq 0$ for <u>all</u> x).

$*$ Special: If $\{y_1, \dots, y_n\}$ = fundamental set $\Rightarrow W(y_1, \dots, y_n)(x) \neq 0$ for <u>all</u> $x \in I$.

(B) n^{th} Order Linear Non-Homogeneous Equations

$$\boxed{a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \dots + a_1(x)y'(x) + a_0(x)y(x) = g(x)} \quad (**)$$

- **Particular Solution** : A particular solution to (**) is any solution y_p that is free of any arbitrary constants.

- **Complementary Solution** : Every non-homogeneous linear equation (**) has an associated homogeneous linear equation (*)

$$\boxed{a_n(x)y^{(n)}(x) + \dots + a_1(x)y'(x) + a_0(x)y(x) = 0} \quad (*)$$

A general solution $y_c = c_1 y_1 + \dots + c_n y_n$ to the homogeneous equation (*) is called a complementary solution (or complementary function).

ANY solution of the non-homogeneous equation (**) is of the form

$$\boxed{\begin{aligned} y &= y_c + y_p \\ &= (c_1 y_1 + \dots + c_n y_n) + y_p \end{aligned}}$$

where y_c is the complementary solution and y_p is any particular solution.

Ex: Consider the equation $\boxed{y'' - 9y = 27}$

Given that the general solution to $\boxed{y'' - 9y = 0}$ is $\boxed{y_c = c_1 e^{3x} + c_2 e^{-3x}}$ (check)

you can easily see that $\boxed{y_p = -3}$ (constant function) is a particular solution.

Then $\boxed{y = c_1 e^{3x} + c_2 e^{-3x} - 3}$ is the general solution to this non-homogeneous equation.

Proof.: Let Y be any solution to (**). Since y_p is also a solution:

$$\begin{aligned} a_n Y^{(n)} + a_{n-1} Y^{(n-1)} + \dots + a_1 Y' + a_0 Y &= g \\ a_n y_p^{(n)} + a_{n-1} y_p^{(n-1)} + \dots + a_1 y_p' + a_0 y_p &= g \end{aligned}$$

$$\ominus \quad a_n (Y - y_p)^{(n)} + \dots + a_1 (Y - y_p)' + a_0 (Y - y_p) = \underline{\underline{0}}$$

$\Rightarrow (Y - y_p)$ = solution to the homogeneous equation

$\Rightarrow (Y - y_p)$ = linear combination of fundamental set

$\Rightarrow Y - y_p = y_c$.