Qualitative Methods; Autonomous Equations

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Math 2552
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Georgia Institute of Technology

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Outline

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  - behavior of solutions near equilibrium points;
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  - monotonicity of solutions.

Direction Fields

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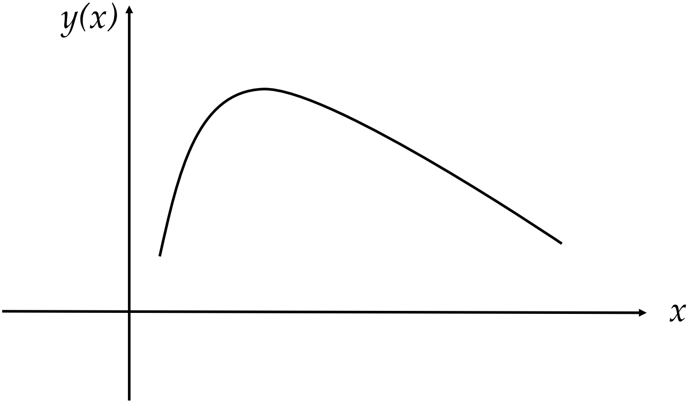

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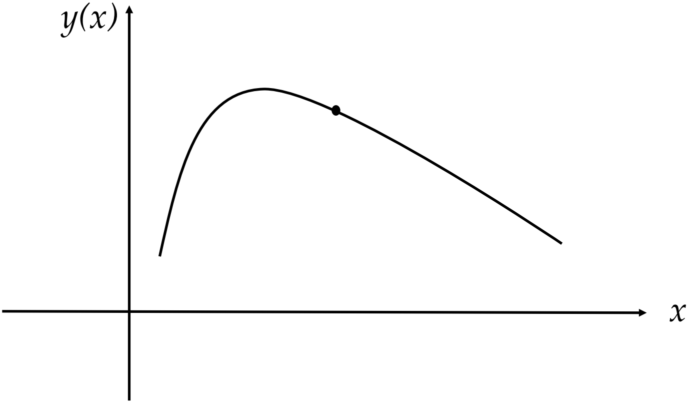

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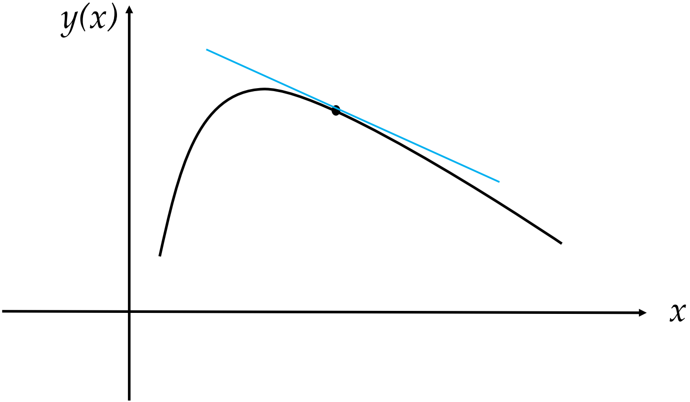

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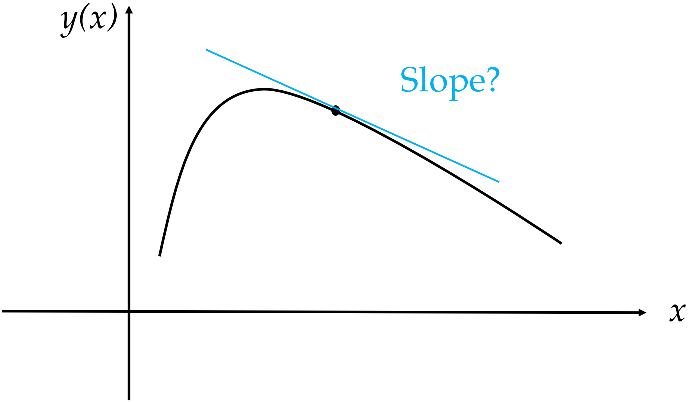

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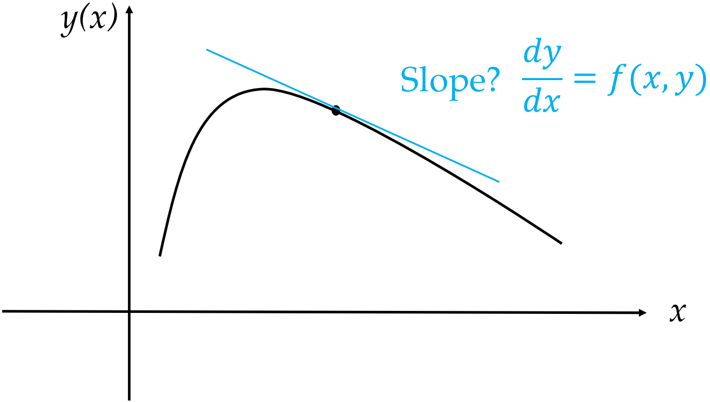

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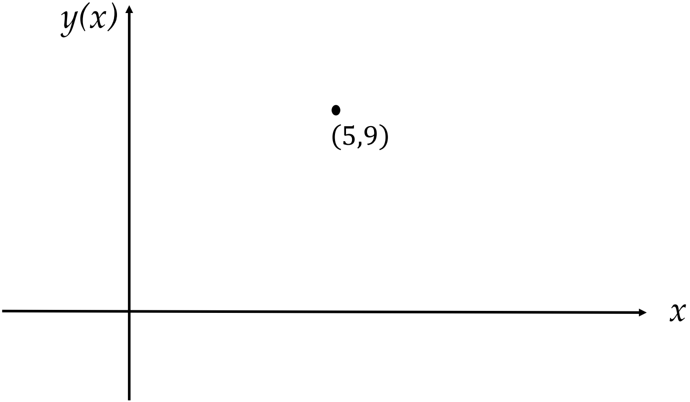

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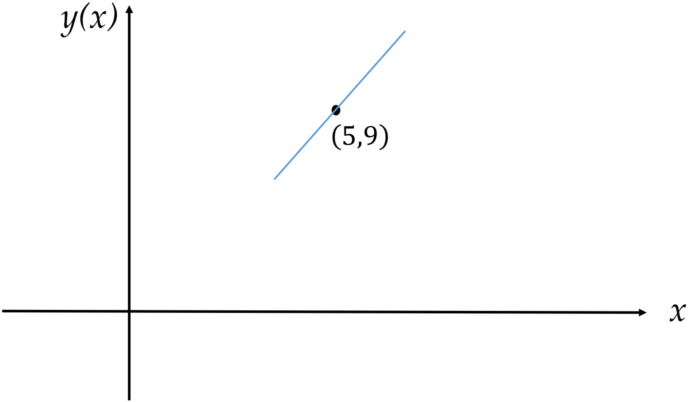

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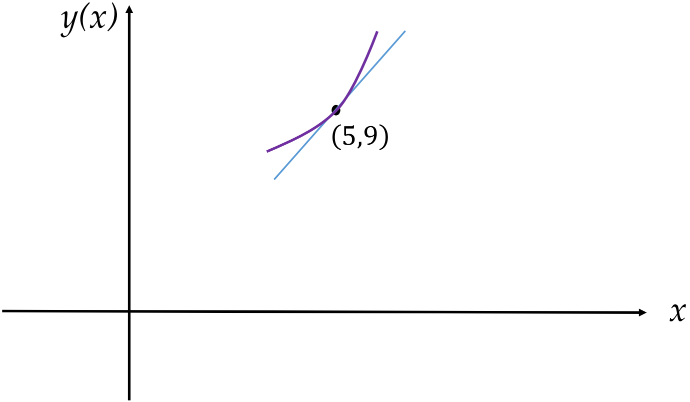
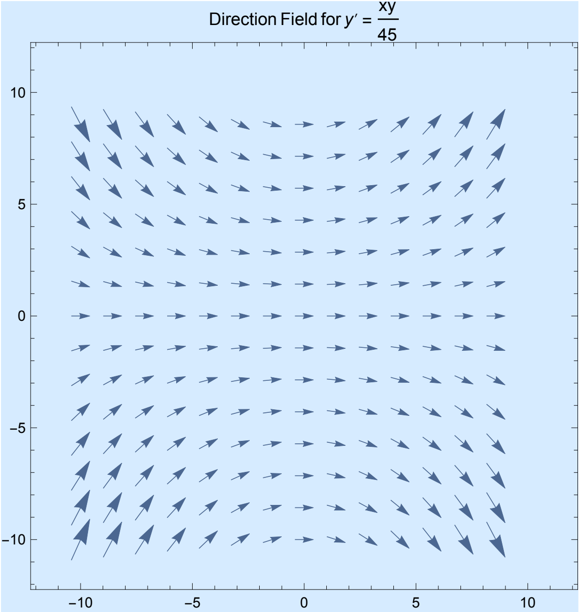
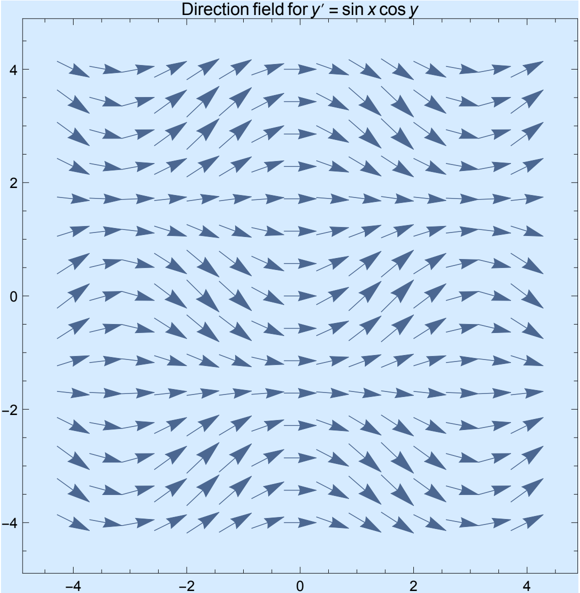
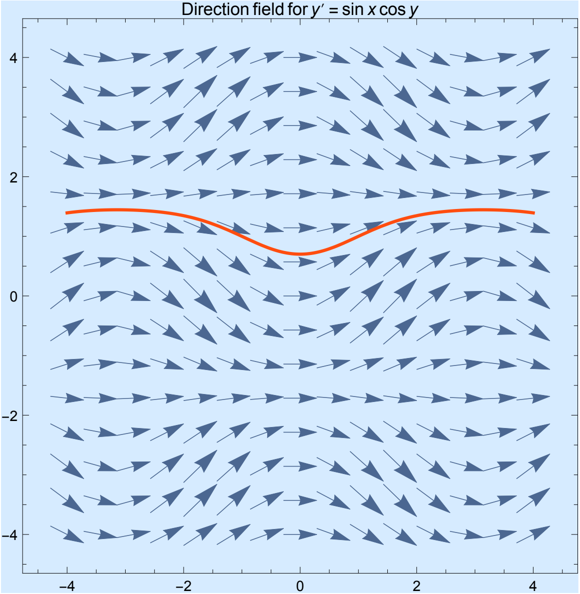



Direction Field for $y' = \frac{xy}{45}$
10
5
0
−5
−10
−10
−5
0
5
10


Direction field for $y' = \sin x \cos y$
4
2
0
−2
−4
−4
−2
0
2
4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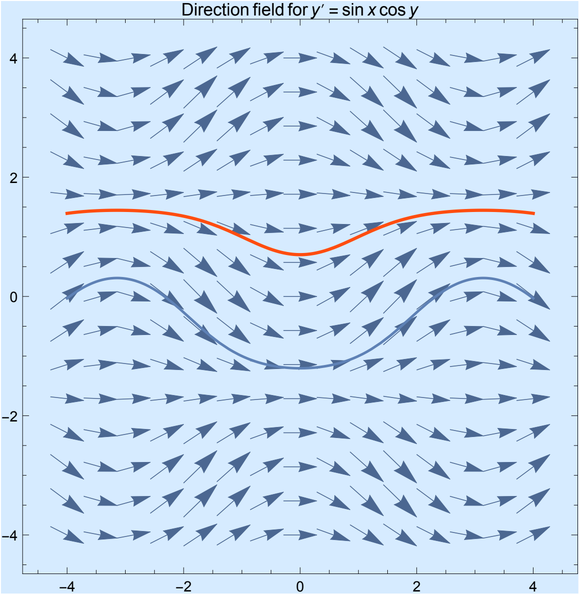
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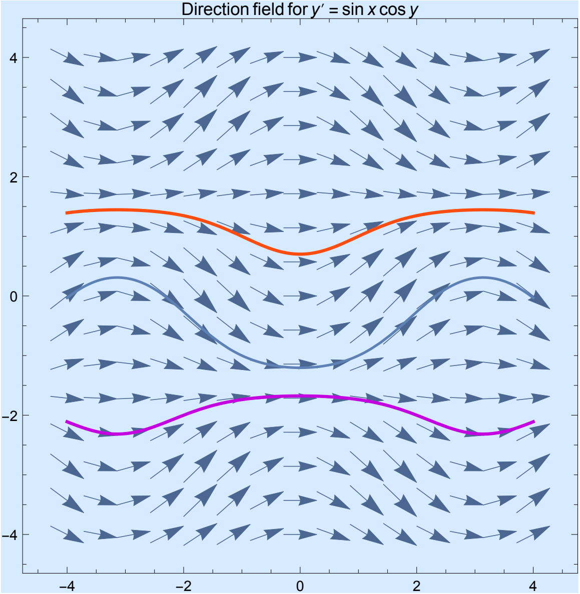
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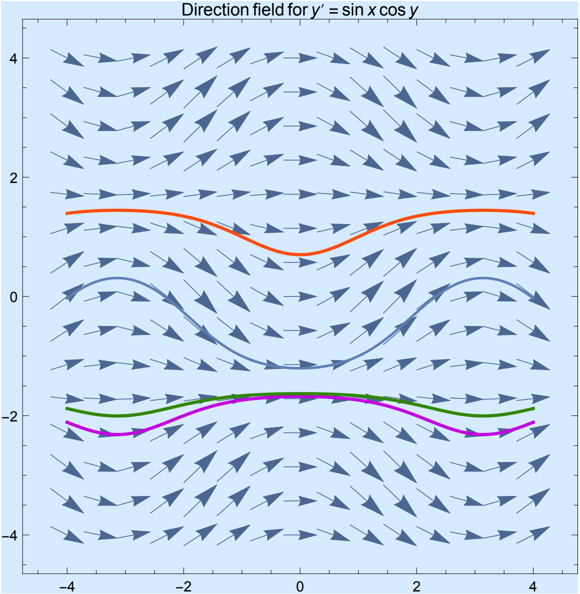
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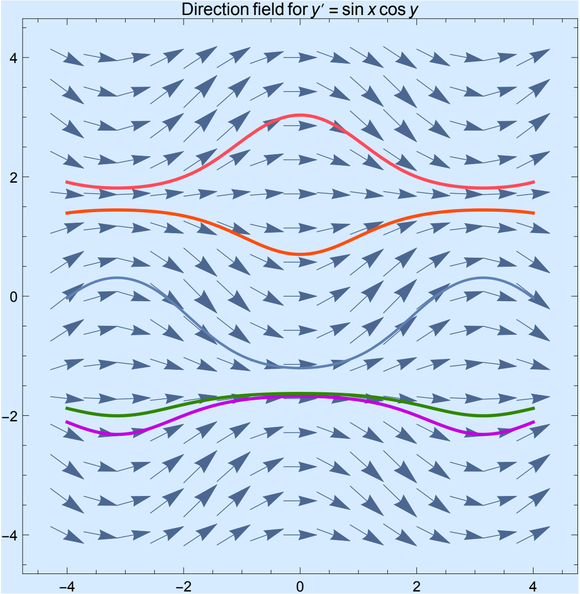
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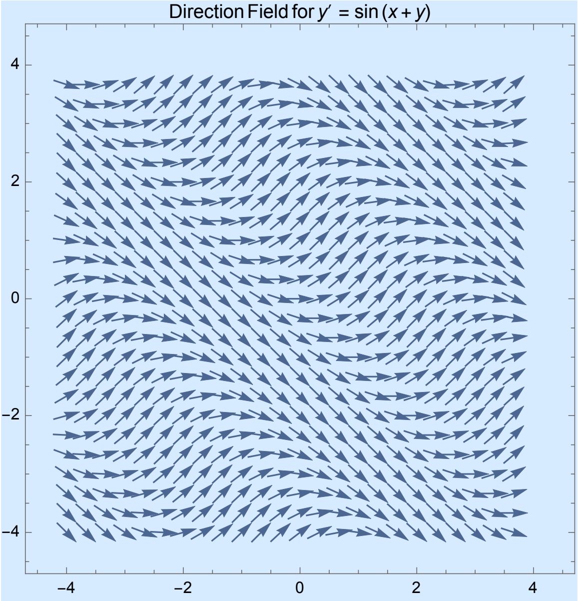
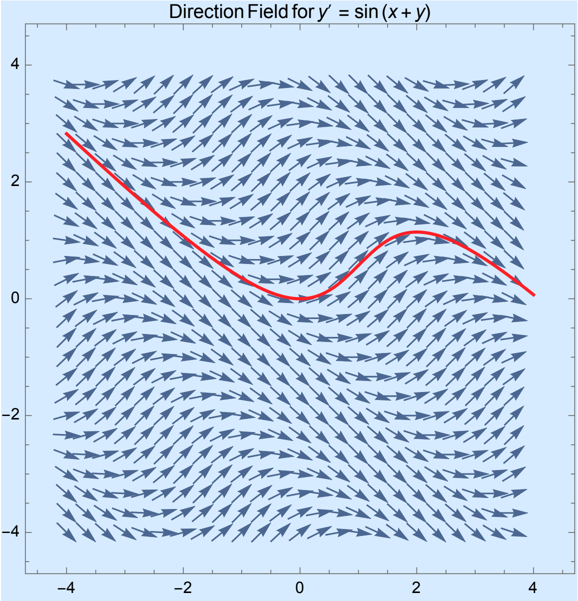

Direction Field for y′ = sin (x + y)
4
2
0
−2
−4
−4
−2
0
2
4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What happens when $f(y) = 0$?

Autonomous Equations

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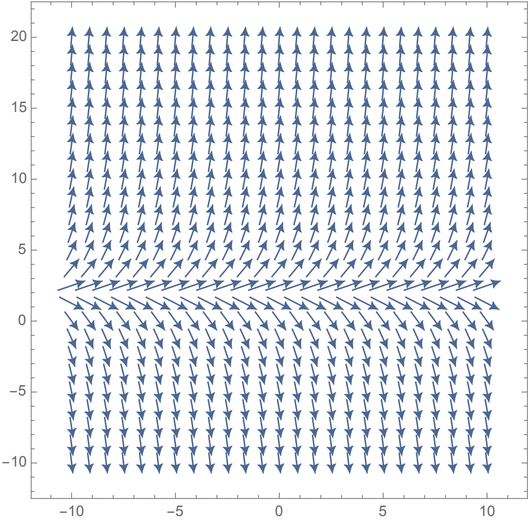

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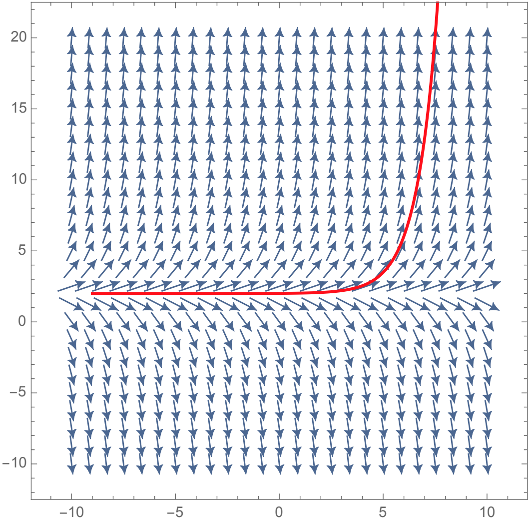

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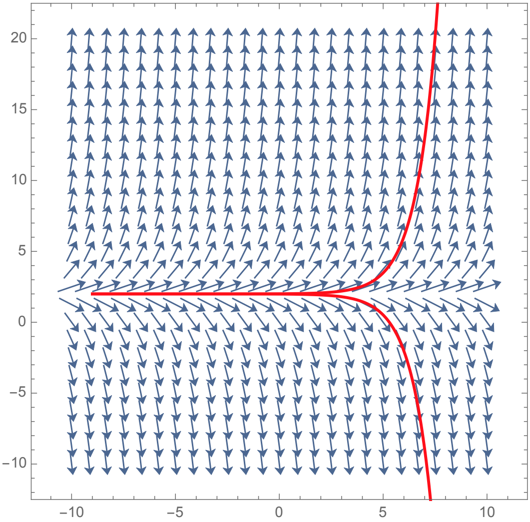

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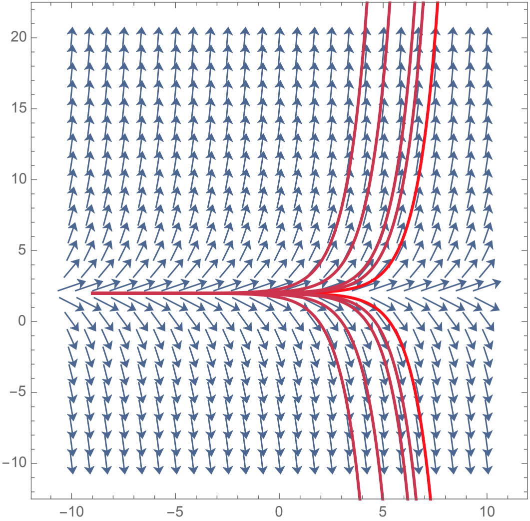

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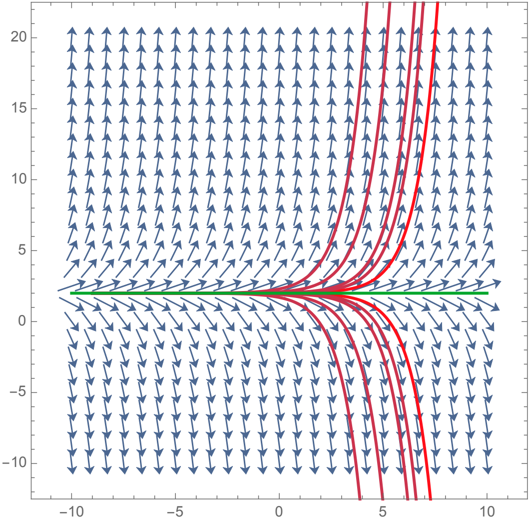

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- If c is a critical point, the constant function

  $$y = c$$

  is a solution to the ODE, called an **equilibrium solution**.

$$\frac{dy}{dx} = f(y)$$

$$f(c) = 0$$

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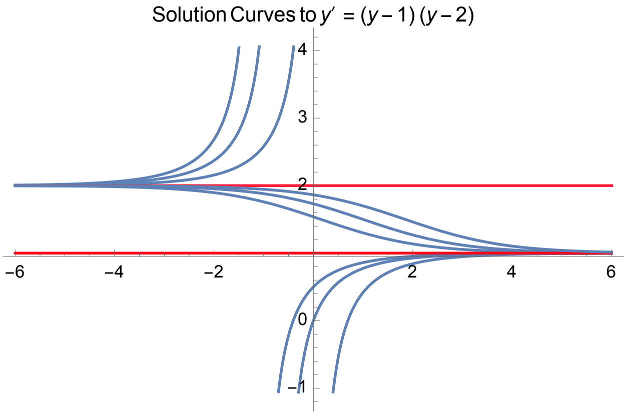

Outline

Look again at

$$\frac{dy}{dx} = y - 2.$$

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Draw a vertical line, representing the y-axis.

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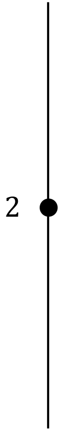
Place the critical points.



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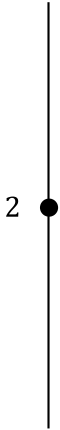
Note: If $y > 2$, then $y' > 0$, so y is increasing.



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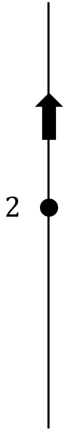

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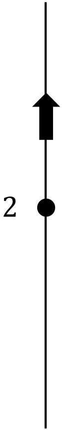

Note: If $y < 2$, then $y' < 0$, so y is decreasing.

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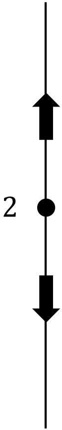

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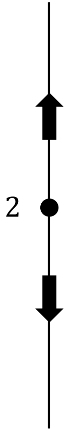
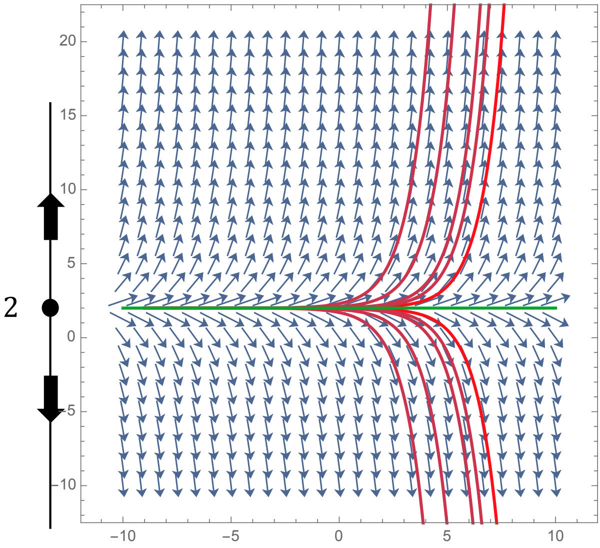
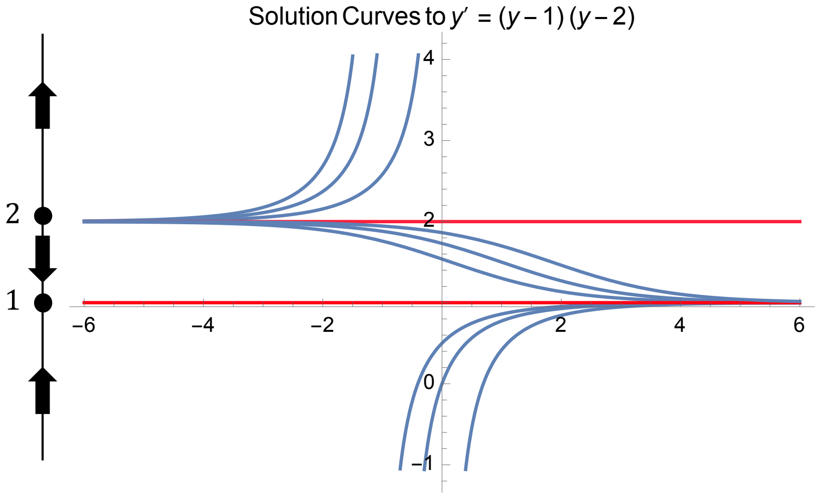
Phase Portrait!

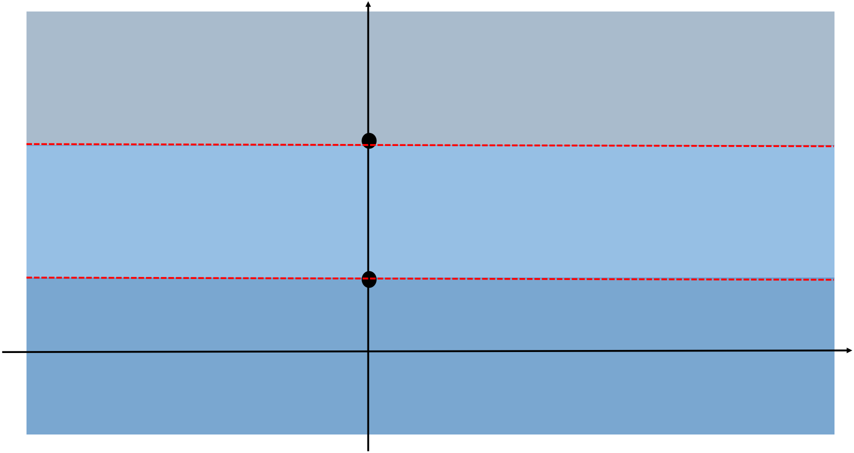



2
20
15
10
5
0
-5
-10
-10
-5
0
5
10

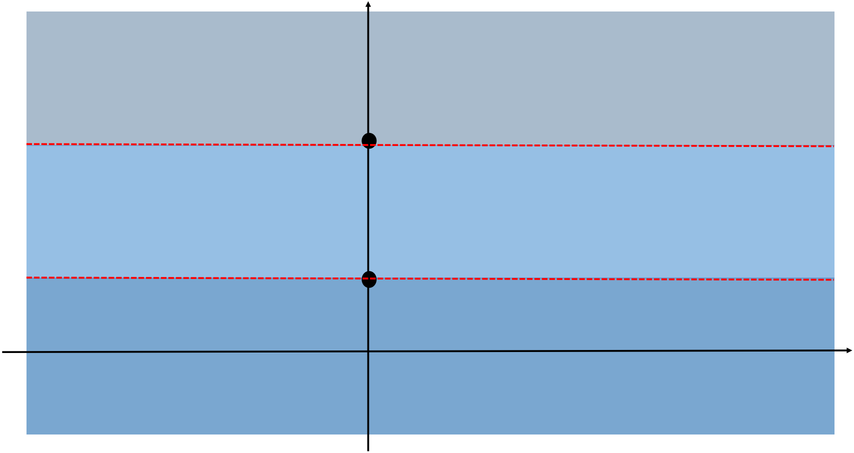

Solution Curves to $y' = (y-1)(y-2)$
2
1
4
3
2
0
−1
−6
−4
−2
2
4
6

Phase Portraits:

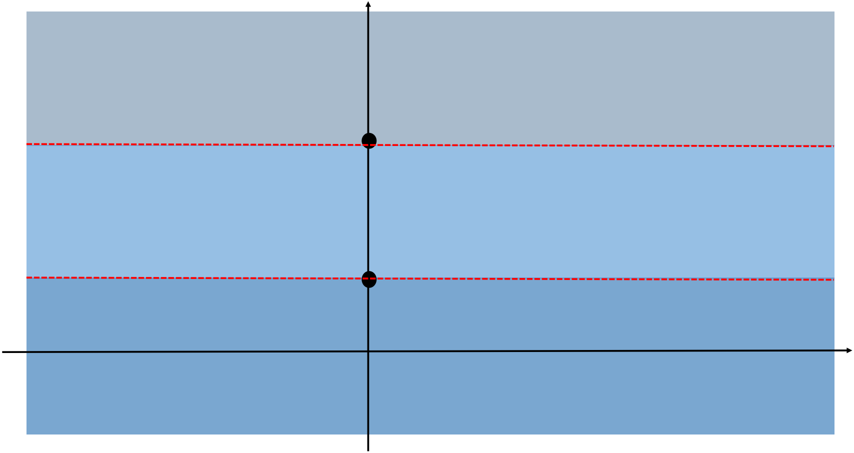
- Draw a vertical line representing the "dependent variable"-axis.
- Place the critical points on the axis. The critical points partition the axis into distinct intervals.
- On each of these intervals, y is *either increasing or decreasing*.
    - **Why?** The critical points are the zeros of the function f. Assuming f is continuous, it must be either positive or negative between two roots.
- Orient each partitioned line segment up or down, if f is increasing or decreasing, respectively.



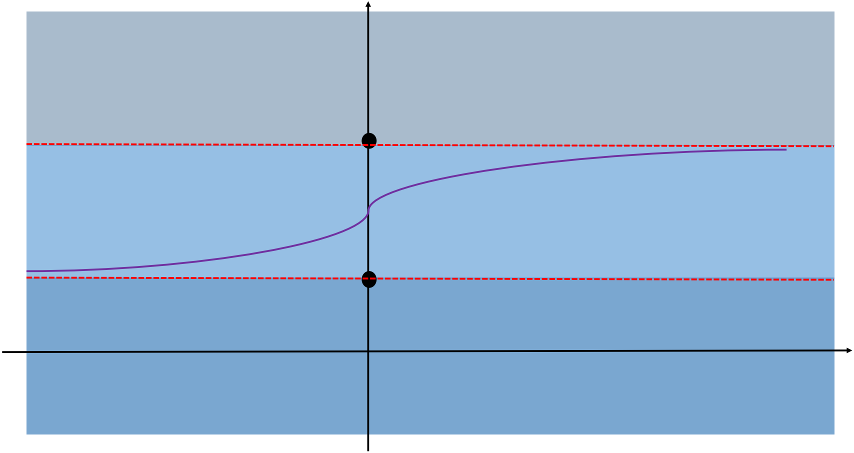
The equilibrium partition the plane into horizontal strips.



What can we learn from this?



(1). Solutions are “trapped” in these regions.



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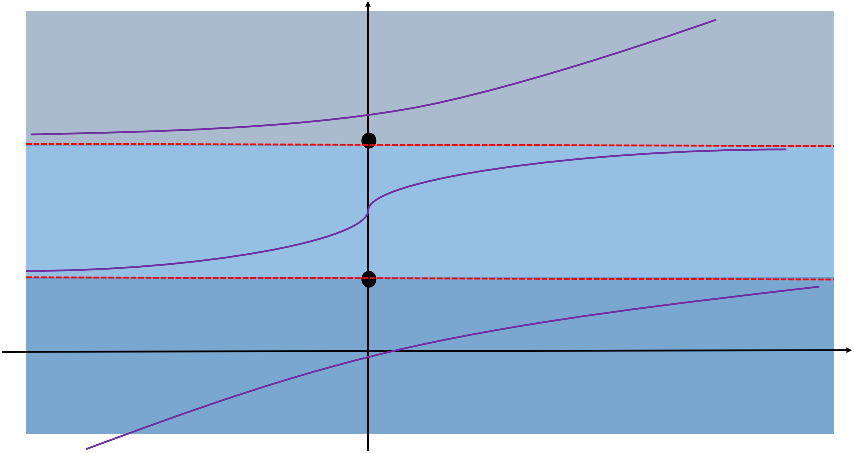

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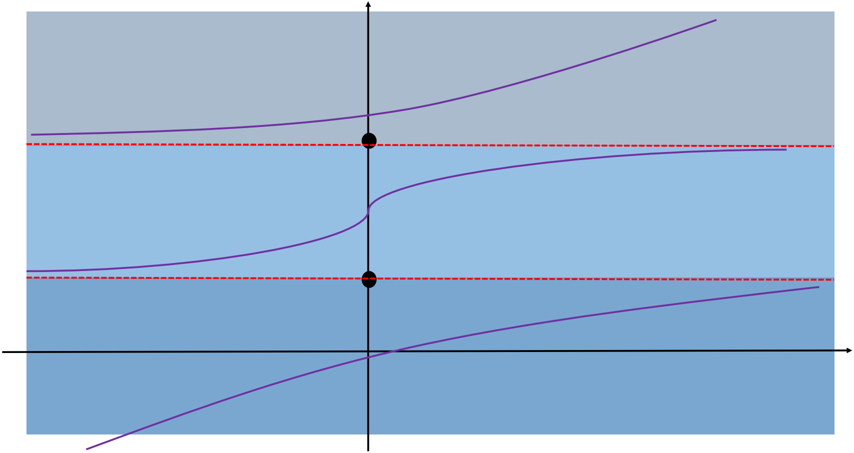
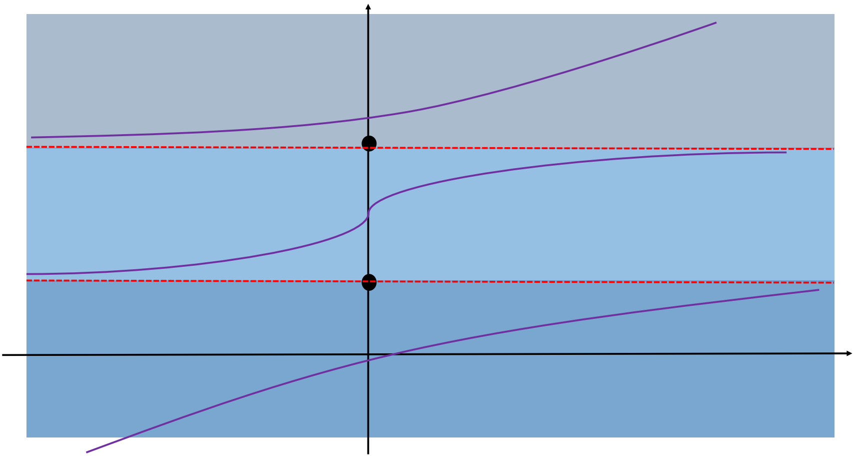


(2). Solutions in these regions are either increasing or decreasing.



(3). Solutions in these regions have the equilibrium solutions as horizontal asymptotes.

Outline

Population Models - Exponential Growth

- $y(t)$ - population of a given species at time t.

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  where r is a non-zero constant.
- $r > 0$: **rate of growth**.
- $r < 0$: **rate of decay**.

Population Models - Exponential Growth

$$\boxed{\frac{dy}{dt} = ry}$$

Solve subject to initial condition

$$y(0) = y_0.$$

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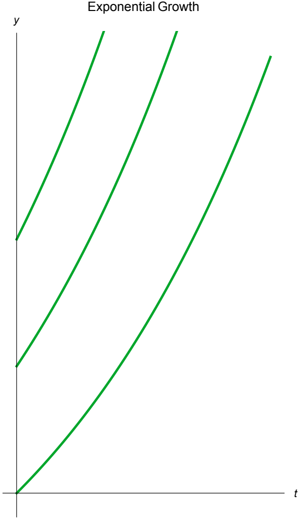

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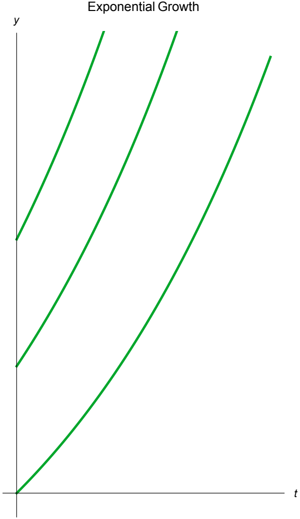

Problem?

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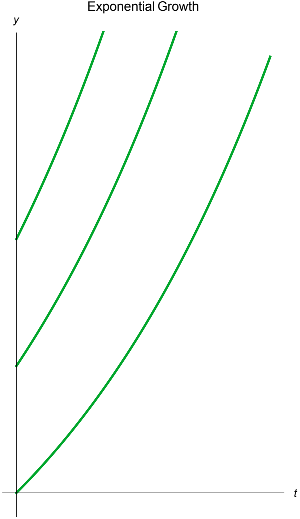

Problem?

This model is fairly accurate under *ideal conditions*, but ideal conditions cannot last forever (depletion of resources).

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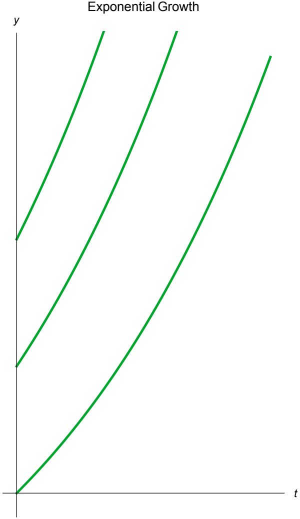

Problem?

This model is fairly accurate under *ideal conditions*, but ideal conditions cannot last forever (depletion of resources).

This model is only accurate for limited periods of time.

Logistic Growth

$$y'(t) = r \cdot y(t)$$

Logistic Growth

$$y'(t) = r \cdot h(y) \cdot y(t)$$

Logistic Growth

$$y' = r\left(1 - \frac{1}{K}y\right)y$$

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K = **carrying capacity** (of the environment).

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The Logistic Equation

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The Logistic Equation

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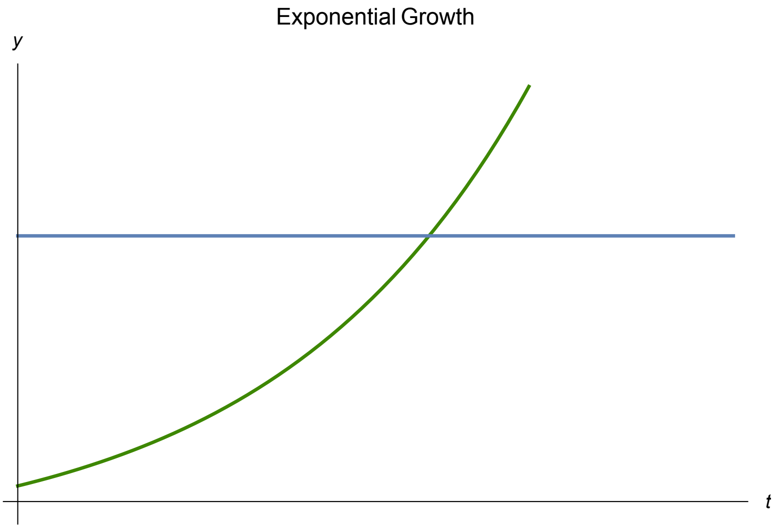
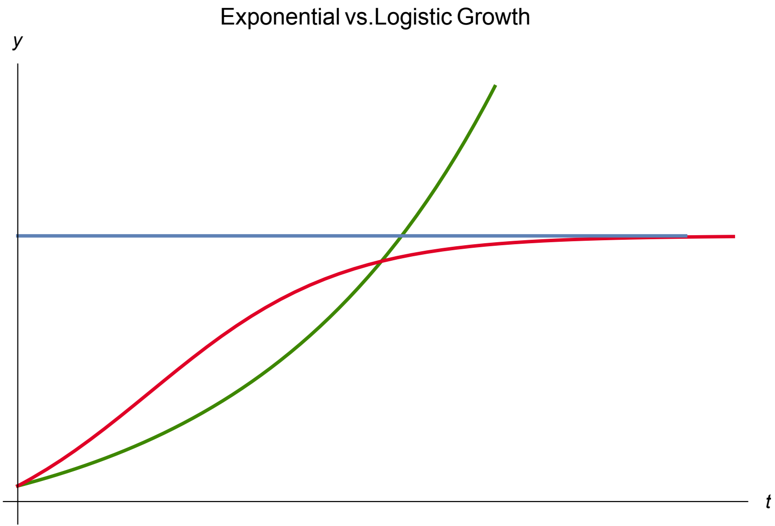
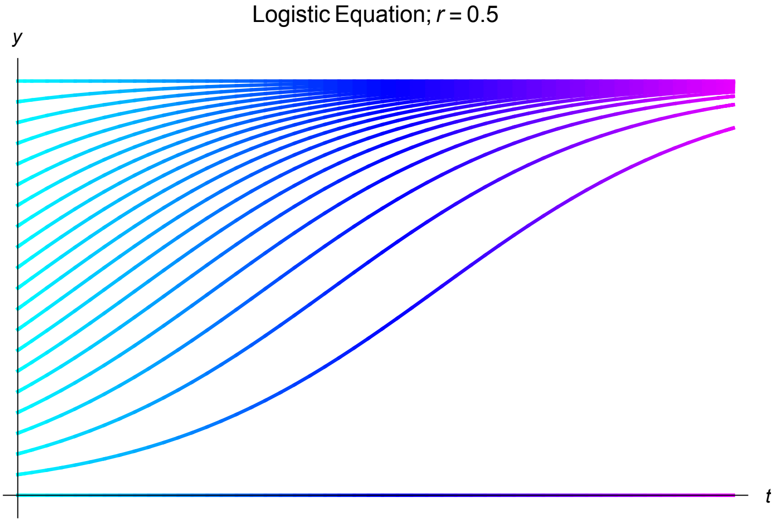
- Autonomous.
- Equilibrium Solutions: $y = 0$; $y = K$.

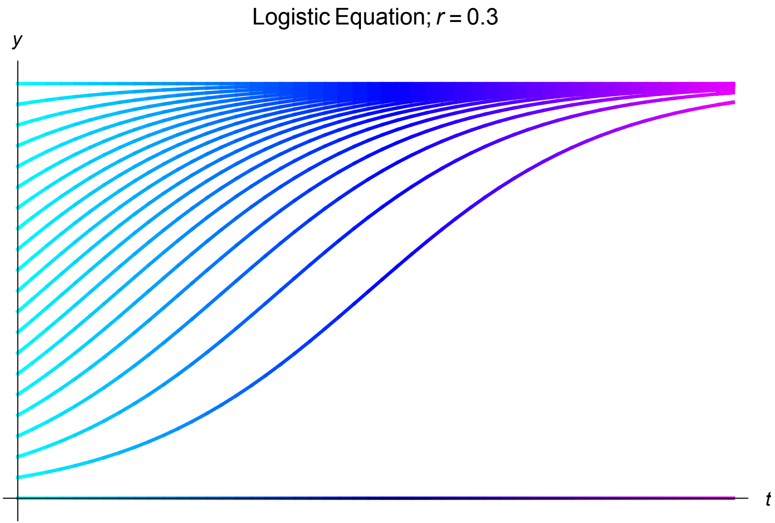
Solution, subject to $y(0) = 0$:

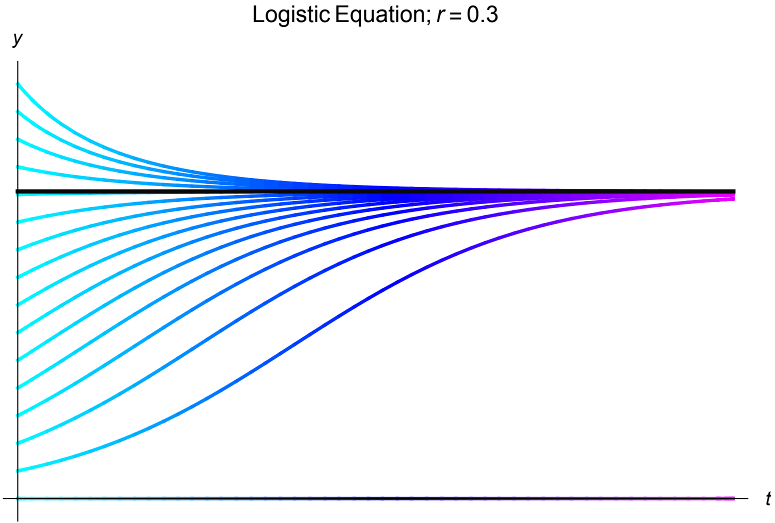
$$\boxed{y(t) = \frac{Ky_0}{y_0 + (K - y_0)e^{-rt}}}$$


Exponential Growth
y
t


Exponential vs.Logistic Growth
y
t


Logistic Equation; $r = 0.5$
y
t


Logistic Equation; $r = 0.3$
y
t


Logistic Equation; $r = 0.3$
y
t