1). Consider the fourth Maclaurin polynomial of $f(x)=e^x$:

$$T_4(x) = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}$$

a). Give an upper bound for the error $R_4(x)$ on $[-4,4]$.

Want to use Taylor's Inequality to approximate $R_4(x)$

$\Rightarrow$ must find an upper bound for $f^{(5)}(x)$.

$$f(x)=e^x$$

$$f^{(5)}(x)=e^x \Rightarrow |f^{(5)}(x)| = e^x \le \overset{\nearrow M}{e^4} \text{ on } x\in[-4,4]$$

so $|x|\le 4$ $\begin{pmatrix} a=0 \\ d=4 \end{pmatrix}$

$$\Rightarrow |R_4(x)| = |f(x)-T_4(x)| \le \frac{e^4}{5!}|x|^5 \text{ for } |x|\le 4.$$

$$\le \frac{e^4}{5!}4^5$$

$$\boxed{|R_4(x)| \le \frac{e^4}{5!}4^5 \text{ for } |x|\le 4}$$

b). Find an interval $[-a,a]$ on which:

$$|R_4(x)| \le 0.001.$$

$|f^{(5)}(x)| = e^x \le e^a$ for $x\in[-a,a]$, or $|x|\le a$.

$\Rightarrow$ by Taylor's Inequality:

$$|R_4(x)| \le \frac{e^a}{5!}|x|^5 \le \frac{e^a}{5!}a^5 \text{ for } |x|\le a.$$

To find an a such that $|R_4(x)| \le 0.001$ on $x\in[-a,a]$, find a such that

$$\frac{e^a}{5!}a^5 \le 0.001 = 10^{-3}$$

Suppose $a<1$ $\Rightarrow e^a < e$

$$\Rightarrow \frac{e^a}{5!}a^5 < \frac{e}{5!}a^5$$

Put $\frac{e}{5!}a^5 \le 0.001 \Rightarrow a \le \sqrt[5]{\frac{5!\cdot 0.001}{e}}$

$$= \left(\frac{12}{100e}\right)^{1/5} (<1 ✓)$$

$$\Rightarrow \boxed{a = \sqrt[5]{\frac{12}{100e}}} \text{ works.}$$