DETERMINE WHETHER OR NOT THE SERIES CONVERGE:

1). $\sum_{k=1}^{\infty} k^2 e^{-k}$

Ratio Test:

$$\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{(k+1)^2 e^{-(k+1)}}{k^2 e^{-k}}\right|$$

$$= \frac{(k+1)^2}{k^2} \, \frac{e^k}{e^{k+1}}$$

$$= \frac{(k+1)^2}{k^2} \, \frac{1}{e} \xrightarrow[k\to\infty]{} \frac{1}{e} < 1$$

=> By the Ratio Test, the series is absolutely convergent.

2). $\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{1}{n^2}\right)$

This is divergent by the Test for Divergence:

$$a_n = (-1)^n \cos\left(\frac{1}{n^2}\right)$$

$$\lim_{n\to\infty} a_n \text{ DNE}$$

(Why? as $n \to \infty$, $\cos\left(\frac{1}{n^2}\right) \to \cos(0) = 1$, but the $(-1)^n$ makes the sequence bounce between (-1) and $(+1)$ indefinitely).

3). $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$

~~Soln. 1~~: Look at $\sum_{n=1}^{\infty} |a_n|$: $\quad a_n = (-1)^n \frac{\ln n}{\sqrt{n}}$

$$|a_n| = \frac{\ln n}{\sqrt{n}}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\ln(n+1)}{\sqrt{n+1}} \, \frac{\sqrt{n}}{\ln n}$$

$$= \frac{\ln(n+1)}{\ln n} \cdot \frac{\sqrt{n}}{\sqrt{n+1}} \xrightarrow[n\to\infty]{} 1 \cdot 1 = 1$$

=> Ratio Test Inconclusive

Alternating Series Test $\quad a_n = (-1)^n b_n, \quad b_n = \frac{\ln n}{\sqrt{n}}$

* $\lim_{n\to\infty} b_n = 0$ ✓ $\quad \left(\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} \overset{\infty/\infty}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = \right.$
* b_n is decreasing ✓ (PROOF:)

Let $f(x) = \frac{\ln x}{\sqrt{x}}$, for $x > 0$.

$$f'(x) = \frac{\frac{1}{x}\sqrt{x} - \ln x \cdot \frac{1}{2\sqrt{x}}}{x} = \frac{2 - \ln x}{2x\sqrt{x}} < 0$$

$f'(x) < 0$ for all $\ln x > 2$, or $x > e^2$

=> $f(x)$ is decreasing on (e^2, ∞) ✓

=> The series is convergent by AST.