FIND RADIUS & INTERVAL OF CONVERGENCE

1). $\displaystyle\sum_{n=0}^{\infty} \frac{x^{n+1}}{3\cdot n!}$

$$a_n = \frac{x^{n+1}}{3\cdot n!}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+2}}{3\cdot(n+1)!}\cdot\frac{3\cdot n!}{x^{n+1}}\right|$$

$$= \lim_{n\to\infty}|x|\cdot\frac{1}{n+1} = \underset{<1}{0} \text{ for all real } x \quad (R=\infty)$$

$\Rightarrow$ By Ratio Test, series converges absolutely for all real x

$$\Rightarrow \boxed{\begin{matrix} R=\infty \\ I=(-\infty,\infty)=\mathbb{R}\end{matrix}}$$

2). $\displaystyle\sum_{n=0}^{\infty} \frac{(x-3)^n}{n^8+1}$

$$\boxed{\begin{matrix} R=1 \\ I=[2,4]\end{matrix}}$$

$$a_n = \frac{(x-3)^n}{n^8+1}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{(n+1)^8+1}\,\frac{n^8+1}{(x-3)^n}\right|$$

$$= \lim_{n\to\infty}|x-3|\,\frac{n^8+1}{(n+1)^8+1} = |x-3|<1$$

$\Rightarrow$ By Ratio Test, series is a.c. for $|x-3|<1$ and divergent for $|x-3|>1$. $\hookrightarrow \Rightarrow R=1$

$|x-3|<1 \Leftrightarrow -1<x-3<1 \Leftrightarrow 2<x<4.$

$x=2 \Rightarrow \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{n^8+1}$ absolutely convergent b/c

$$\sum_{n=0}^{\infty}\left|\frac{(-1)^n}{n^8+1}\right| = \sum_{n=0}^{\infty}\frac{1}{n^8+1} = 1+\sum_{n=1}^{\infty}\frac{1}{n^8+1}$$

$$\le 1+\underbrace{\sum_{n=1}^{\infty}\frac{1}{n^8}}_{\text{conv. p-series}}<\infty$$

$\hookrightarrow$ or convergent by AST

$x=4 \Rightarrow \displaystyle\sum_{n=0}^{\infty}\frac{1}{n^8+1}$ convergent

(by comparison to $\displaystyle\sum \frac{1}{n^8}<\infty$ conv. p-series)

3). $\displaystyle\sum_{n=1}^{\infty} \frac{3^n\cdot(x+3)^n}{\sqrt{n}}$

This is centered at $x=-3$ so look for $|x+3|<R$

$$a_n = \frac{3^n\cdot(x+3)^n}{\sqrt{n}}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}(x+3)^{n+1}}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{3^n(x+3)^n}\right|$$

$$= \lim_{n\to\infty} 3|x+3|\,\frac{\sqrt{n}}{\sqrt{n+1}} = 3|x+3|<1$$

$\Rightarrow$ By Ratio Test, the series is abs. conv. for

$3|x+3|<1 \Rightarrow |x+3|<\frac{1}{3}$ $\quad R=1/3$

and divergent for $|x+3|>\frac{1}{3}$.

Endpoints? $\quad -\frac{1}{3}<x+3<\frac{1}{3} \Leftrightarrow -\frac{10}{3}<x<-\frac{8}{3}$

$x=-\frac{10}{3}$: $\displaystyle\sum_{n=1}^{\infty}\frac{3^n\cdot\left(-\frac{10}{3}+3\right)^n}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{3^n\cdot\frac{(-1)^n}{3^n}}{\sqrt{n}}$

$$= \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}} \text{ convergent by AST}$$

$x=-\frac{8}{3}$: $\displaystyle\sum_{n=1}^{\infty}\frac{3^n\cdot\left(-\frac{8}{3}+3\right)^n}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{3^n\cdot\frac{1}{3^n}}{\sqrt{n}}$

$$= \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}} \text{ divergent p-series w/ } p=\tfrac{1}{2}$$

$$\boxed{\begin{matrix} R=\frac{1}{3} \\ I=\left[-\frac{10}{3},-\frac{8}{3}\right)\end{matrix}}$$