1). $\sum_{n=1}^{\infty}(-1)^n \frac{1}{4n-1} = \sum_{n=1}^{\infty} a_n$, where $a_n = (-1)^n \frac{1}{4n-1}$

a) Apply the Comparison Test to $\sum_{n=1}^{\infty} |a_n|$.
Explain why this strategy fails in this case.

$$\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\frac{1}{4n-1}$$

$\frac{1}{4n-1} > \frac{1}{4n}$ and $\sum_{n=1}^{\infty}\frac{1}{4n} = \infty$ (harmonic series $\times \frac{1}{4}$)

$\Rightarrow$ By the Comparison Test, $\sum_{n=1}^{\infty}|a_n| = \infty$ (divergent)

$\Rightarrow$ $\boxed{\sum_{n=1}^{\infty} a_n \text{ is } \underline{\text{not}} \text{ absolutely convergent.}}$

$\rightarrow$ This statement implies nothing about whether or not $\sum_{n=1}^{\infty} a_n$ converges or not.

Remember: Absolutely conv. $\Rightarrow$ Convergent
but Not Absolutely conv. $\Rightarrow$?!

b) Use an appropriate test to determine convergence.

Alternating Series Test: $b_n = \frac{1}{4n-1}$

* b_n is decreasing: $b_{n+1} = \frac{1}{4n+3} \leq \frac{1}{4n-1} = b_n$ ✓
* $\lim_{n\to\infty} b_n = 0$ ✓

$\Rightarrow$ by AST, $\sum_{n=1}^{\infty} a_n$ converges.

2). $\sum_{n=1}^{\infty}(-1)^n \frac{1}{4n^2+1}$

a). Apply the Comparison Test to $\sum_{n=1}^{\infty}|a_n|$.
Does this strategy work here?

$$\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\frac{1}{4n^2+1}$$

$\frac{1}{4n^2+1} < \frac{1}{4n^2}$

$\sum_{n=1}^{\infty}\frac{1}{4n^2}$ is convergent ($\frac{1}{4}\times$ p-series w/ $p=2$)

$\Rightarrow$ by C.T., $\sum_{n=1}^{\infty}|a_n|$ converges

$\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ is absolutely convergent $\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ is convergent
(Strategy works).

b). Can you use the Comparison Test on $\sum_{n=1}^{\infty} a_n$ directly for either #1 or #2? Explain why or why not.

No. Because the Comparison Test applies only to series w/ positive terms.

Moral of the story: Say you have some series $\sum_{n=1}^{\infty} a_n$ and not all terms of $\{a_n\}$ are positive. Then you may only apply :
- The Integral Test;
- The (Limit) Comparison Test;

to $\sum_{n=1}^{\infty}|a_n|$, but not to $\sum_{n=1}^{\infty} a_n$ directly. If you do, and obtain:

* $\sum_{n=1}^{\infty}|a_n|$ is convergent $\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ is convergent. ☺
* $\sum_{n=1}^{\infty}|a_n|$ diverges $\Rightarrow$?! no conclusion about $\sum_{n=1}^{\infty} a_n$.

(Do smth else – AST? T for D)