§. Qualitative Methods; Autonomous Equations

- **Qualitative Methods**: determine properties of the solutions of a DE using geometrical methods (without actually solving the DE).
  - Examples of properties investigated via qualitative methods:
    - equilibrium solutions
    - behavior of solutions near equilibrium points
    - long-term behavior of solutions (limits at $\pm\infty$)
    - monotonicity of solutions.

- **Direction Fields**: Consider a first order ODE: $\frac{dy}{dx} = f(x,y)$

Suppose $y(x)$ is a solution of this ODE on some interval $I \subset \mathbb{R}$.
Then $y(x)$ must be differentiable, and therefore continuous on I

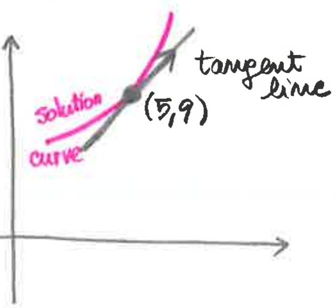
$\Rightarrow$ at every point $(x, y(x))$, the curve has a tangent line whose slope is exactly $\frac{dy}{dx}(x) = f(x, y(x))$

$\Rightarrow$ $\frac{dy}{dx} = f(x,y) \longrightarrow$ slope of tangent line to the graph of a solution $y(x)$ passing through $(x, y(x))$.

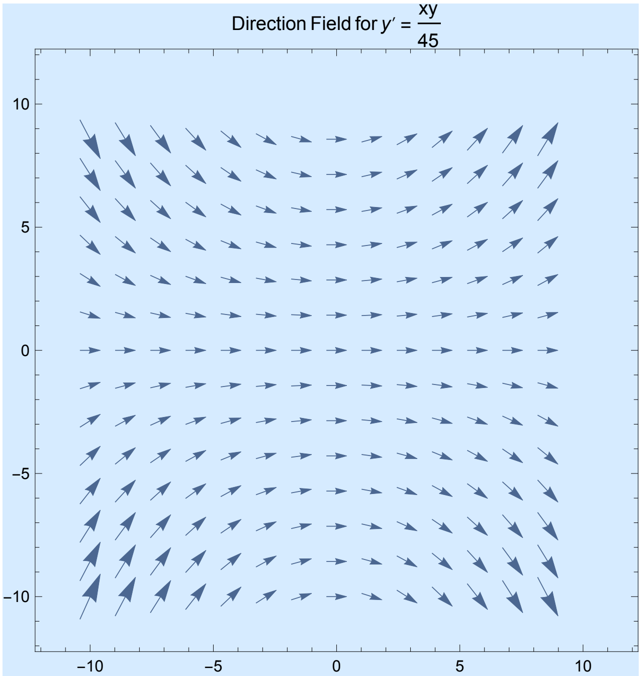
Ex: $\frac{dy}{dx} = \frac{1}{45}xy$

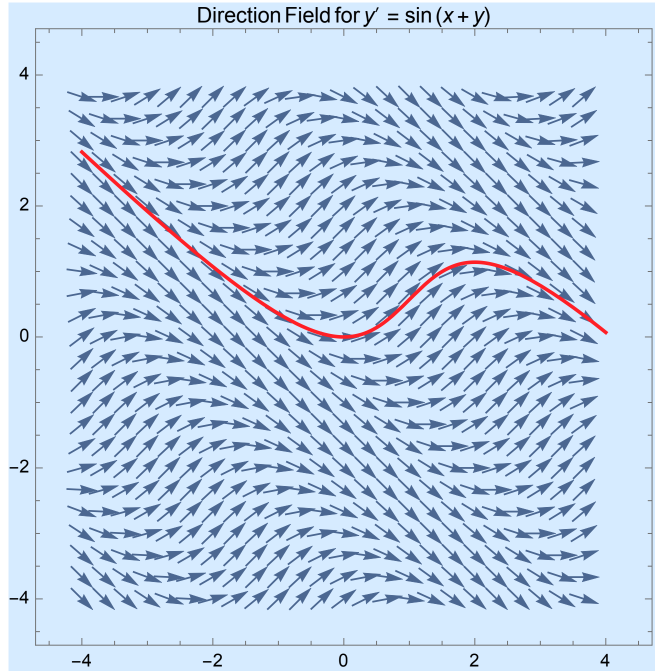
$x = 5, y = 9 \Rightarrow \left.\frac{dy}{dx}\right|_{(5,9)} = 1$

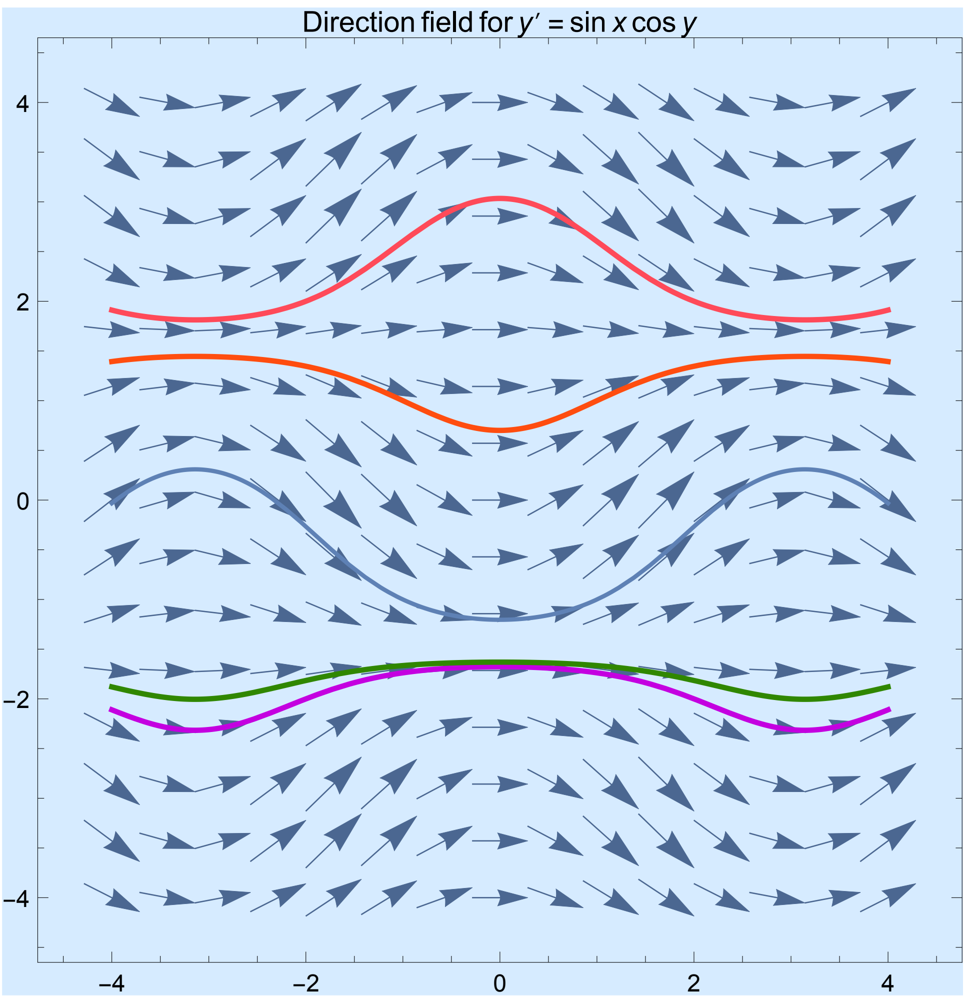
$\Rightarrow$ tangent line at $(5,9)$ has slope 1.



- Doing this systematically, i.e. evaluate $f(x,y)$ over a rectangular grid of points (x,y) and draw a small line element w/ the corresponding slope, oriented in the direction of increasing x, yields a **direction field**.







Direction Fields give us a geometrical sense of the solutions, since any solution curve must follow the flow of the field.

- **Automomous Equations**: An automomous first order ODE is an equation of the form:

$$\boxed{\frac{dy}{dx} = f(y)} \qquad (*)$$

(no independent variable on the right side).

- For example: $\frac{dy}{dx} = y^2 + \sin(y)$ (automomous) ; $\frac{dy}{dx} = xy$ (not automomous).

- An automomous equation is separable: if $f(y) \neq 0$, we can write the eqn. as

$$\frac{1}{f(y)} dy = dx$$

and solve by integrating both sides.

- What if $f(y) = 0$?
  - Any number $c \in \mathbb{R}$ such that $f(c) = 0$ is called a critical point (aka equilibrium point or stationary point) of the ODE $(*)$.
  - The constant functions $y = c$, where c is any equilibrium point are the (only) constant solutions to the ODE$(*)$.
    To see this, let $y(x) = c$ for all x, where $f(c) = 0$. Then $\frac{dy}{dx} = f(c) = 0$, so $y = c$ is a solution to $(*)$.
  - The constant solutions $y = c$, where c is an equilibrium point, are called the equilibrium solutions (because they correspond to the situations where y does not vary as x increases or decreases).

Ex $\frac{dy}{dx} = y - 2$

- $y - 2 = 0 \Leftrightarrow \boxed{y = 2}$ Equilibrium solution
- To find the other solutions:

$$\int \frac{1}{y-2} dy = \int dx$$

$$\ln|y-2| + c = x$$

$$c(y-2) = e^x$$

$$y - 2 = ce^x$$

$$\boxed{y = ce^x + 2}$$

Ex: $\frac{dy}{dx} = (y-1)(y-2)$

- $(y-1)(y-2) = 0 \Rightarrow \boxed{y = 1}$ & $\boxed{y = 2}$ Equilibrium solutions
- The other solutions:

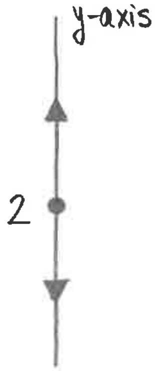
$$\int \frac{1}{(y-1)(y-2)} dy = \int dx$$

$$\ln\left|\frac{y-2}{y-1}\right| = x + c$$

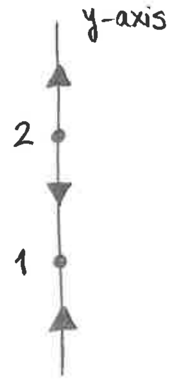
$$\boxed{\frac{y-2}{y-1} = ce^x}$$

Phase Portraits

- Look again at the autonomous equation $\frac{dy}{dx} = y-2$.
  - If $y<2$, then $y' = y-2 < 0$, which tells us that any solution y will be <u>decreasing</u> in the region $y<2$.
  - If $y>2$, then $y' = y-2 > 0$, so a solution y will be <u>increasing</u> when $y>2$.



Phase portrait for $y' = y-2$.



Phase portrait for $y' = (y-1)(y-2)$

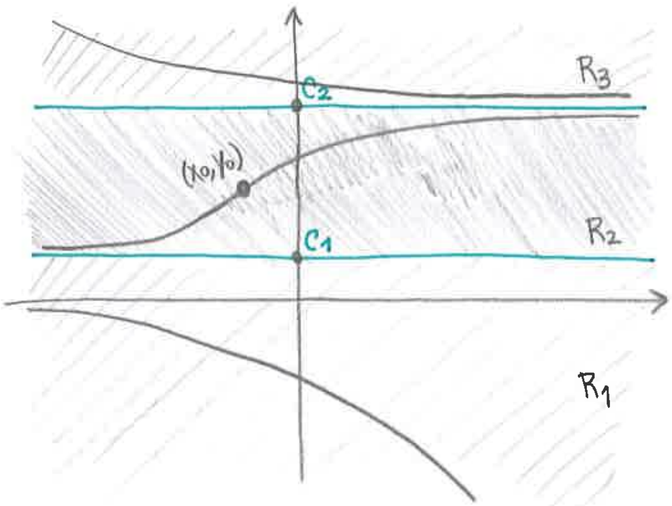
- Similarly, $y' = (y-1)(y-2)$ has: any solution is <u>increasing</u> when $(y-1)(y-2) > 0$, or $y \in (-\infty, 1) \cup (2, \infty)$ and decreasing when $(y-1)(y-2) < 0$, or $y \in (1,2)$.

- <u>Generally</u>: <u>The phase portrait of an autonomous ODE</u> $\boxed{y' = f(y)}$ (assume f & f' are continuous on some interval)
  - Draw a vertical line representing the y-axis (to be read: "dep. var."-axis) This is called the **phase line**.
  - Draw the critical points on this line. The critical points <u>partition</u> the phase line into distinct intervals.
  - On each interval partitioned by critical points, any solution y will be <u>either</u> <u>increasing</u> or <u>decreasing</u>.

orient the line ↑ orient the line ↓

Why? Recall that the critical/equilibrium points are the zeros of the function f, and so f will be either ⊕ or ⊖ between critical points. And, f is essentially the derivative of y.

- What can we learn from a phase portrait about the solutions to an autonomous ODE $y' = f(y)$? Suppose for simplicity that f and f' are both continuous functions of y. Then the hypotheses of the Existence & Uniqueness Theorem are satisfied, so through any point (x_0, y_0) there passes exactly one solution curve!



The equilibrium solutions partition the plane into horizontal strips: $R_1, R_2, \dots, R_n$. These strips reveal some important features of the solutions to the ODE:

(1) Solutions are "trapped" in these regions:

- If a point (x_0, y_0) belongs to a region R_i and $y(x)$ is a solution passing through this point, then $y(x)$ remains in the region R_i for all x

(This is because two solution curves cannot intersect, and for $y(x)$ to leave the region, it would have to cross one of the equilibrium solutions).

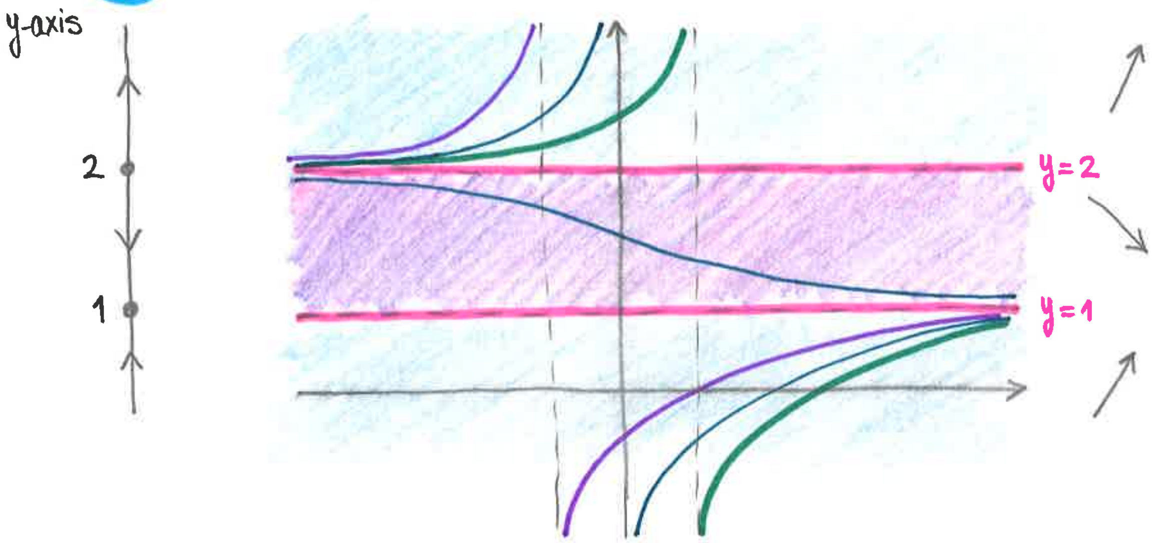
(2) Solutions are either increasing or decreasing in a region:

- This is because $f(y)$, assumed continuous, can be either positive or negative between two of its zeros (the critical points).

(3) The equilibrium solutions are horizontal asymptotes for the other solutions:

- If $y(x)$ is a solution trapped in one of the regions, and c is a critical point bounding the region (either from above or below) then $y(x)$ must approach c either as $x \to -\infty$ or $x \to \infty$.

Ex $\frac{dy}{dx} = (y-1)(y-2)$



- All the solutions in $[y>2]$ are increasing $\Rightarrow$ must have $\lim_{x\to -\infty} y(x) = 2$
- All solutions in $[-2<y<2]$ are decreasing $\Rightarrow$ must have $\lim_{x\to -\infty} y(x) = 2$ & $\lim_{x\to \infty} y(x) = 1$
- All solutions in $[y<2]$ are increasing $\Rightarrow$ must have $\lim_{x\to \infty} y(x) = 1$

Solutions to this ODE are: $\frac{y-2}{y-1} = ce^x$, or $\boxed{y = \frac{ce^x-2}{ce^x-1}}$ could have vertical asymptotes

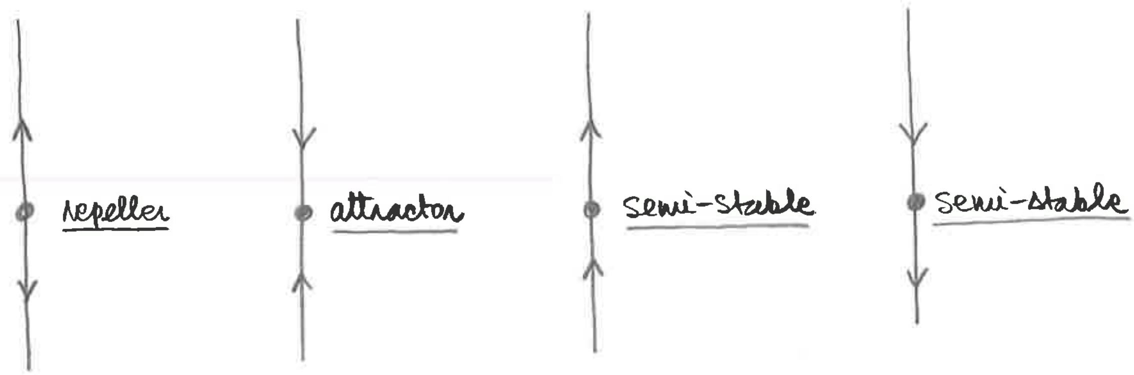
$ce^x - 1 = 0 \Rightarrow e^x = 1/c$ (only possible when $c>0$)

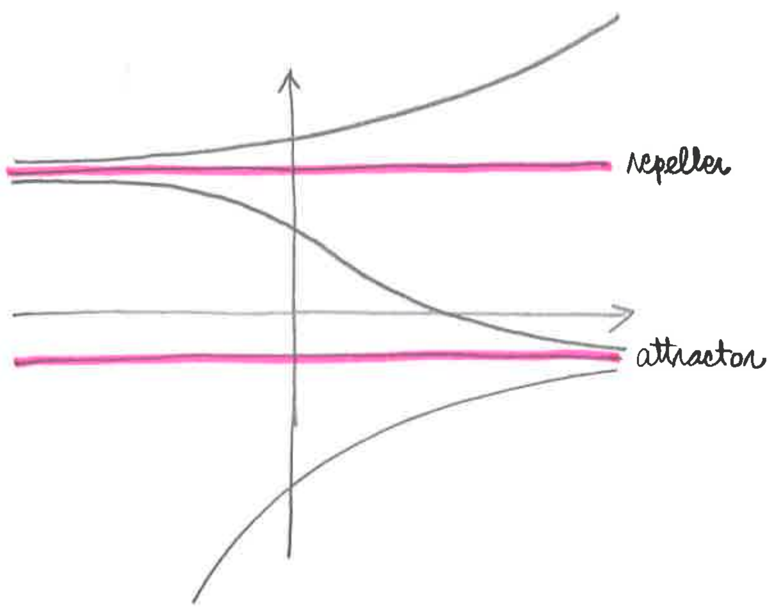
$\Rightarrow$ all solutions $y = \frac{ce^x-2}{ce^x-1}$ with $c>0$ have vertical asymptotes at $x = \ln(\frac{1}{c})$
(these are the solutions in the blue regions above)

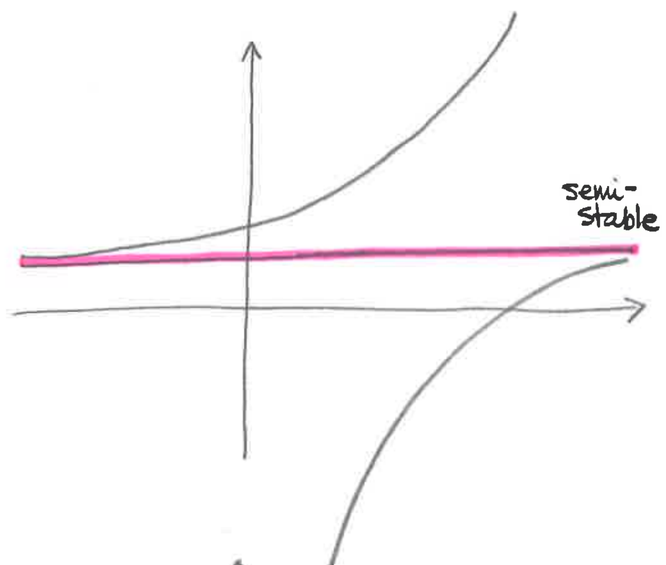
$\Rightarrow$ all solutions $y = \frac{ce^x-2}{ce^x-1}$ with $c<0$ have no vertical asymptotes
(these are the solutions in the purple region above)

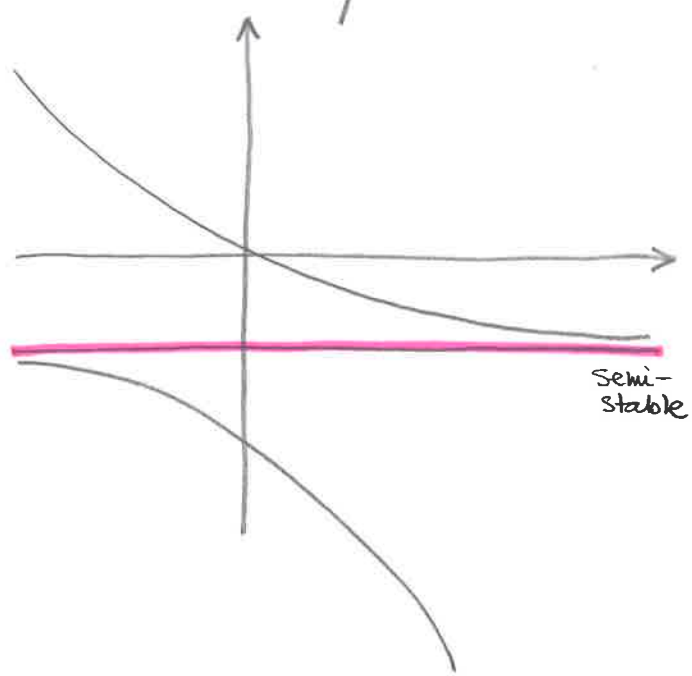
Attractors & Repellers

Four possible behaviors around a critical point in a phase portrait:









* Repeller / Asymptotically Unstable:
  Any solution passing close enough to $y=c$ moves away from c as $x \to \infty$

* Attractor / Asymptotically Stable:
  Any solution passing close enough to $y=c$ moves towards c as $x \to \infty$

* Semi-Stable:
  Behaves like an attractor on one side and like a repeller on the other side.

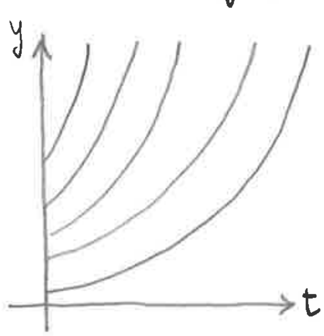
Population Models

- **Exponential Growth**: Let $y(t)$ denote the population of a given species at time t.
  - Simplest hypothesis: the rate of change of y is proportional to the current value of y (if the population doubles, the number of births also doubles):

$$\boxed{\frac{dy}{dt} = ry}$$

where r is a non-zero constant called rate of growth if $r>0$, rate of decay if $r<0$.

  - Solve subject to initial condition $y(0)=y_0$

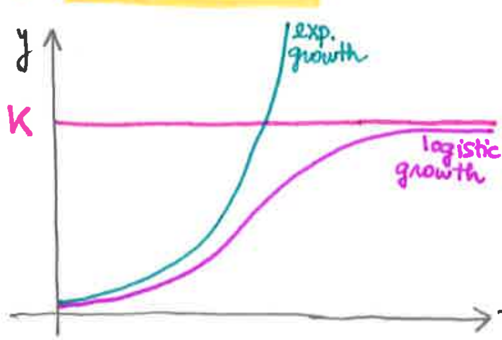


$$\left.\begin{array}{l}\frac{1}{y}dy = r\,dt \Rightarrow \ln|y| = rt + c \Rightarrow y = ce^{rt} \\ t=0, y=y_0 \Rightarrow y_0 = c\end{array}\right\} \Rightarrow \boxed{y(t) = y_0 e^{rt}}$$

  - If $r>0$, this is exponential growth (exp. decay if $r<0$).
    - * reasonably accurate for many populations, under ideal conditions.
    - * problem: The ideal conditions cannot last forever (resources deplete)

    $\Rightarrow$ usually accurate only for limited periods of time.

- **Logistic Growth**: In other words, there is some number K, called the carrying capacity (of the environment) that bounds the population of the species.



$$\boxed{\frac{dy}{dt} = r\left(1-\frac{1}{K}y\right)y}$$

$\rightarrow$ Logistic equation (aka Verhulst equation)

(K = maximum possible population)

* Remark: when the population is very small compared to K, the two models are basically indistinguishable. But, as the population grows:

- the exponential model grows indefinitely;
- the logistic model approaches K and the growth rate approaches 0.

- Solve the logistic equation subject to $y(0)=y_0$:

$$\boxed{\frac{dy}{dt}=r\left(1-\frac{1}{K}y\right)y}$$

$\longrightarrow$ autonomous equation

$\longrightarrow$ equilibrium solutions: $\boxed{y=0;\ y=K}$

$$\int \frac{1}{y\left(1-\frac{1}{K}y\right)}\,dy=\int r\,dt \Rightarrow \ln\left|\frac{y}{y-1/K\,y}\right|=rt \Rightarrow \frac{y}{y-1/K\,y}=ce^{rt} \quad (c\neq 0)$$

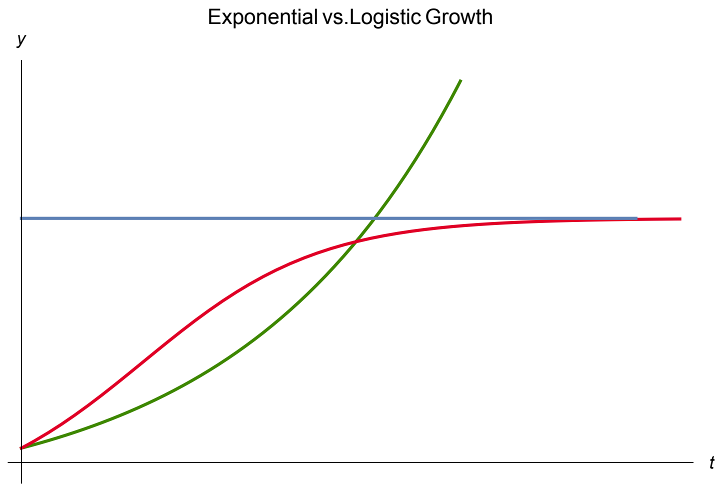
$$\Rightarrow y=\frac{ce^{rt}}{1+\frac{1}{K}ce^{rt}}=\frac{1}{\frac{1}{K}+\frac{1}{c}e^{-rt}}=\frac{1}{\frac{1}{K}+ce^{-rt}}$$

$$y(0)=y_0 \Rightarrow y_0=\frac{1}{\frac{1}{K}+c} \Rightarrow c=\frac{K-y_0}{y_0K}$$

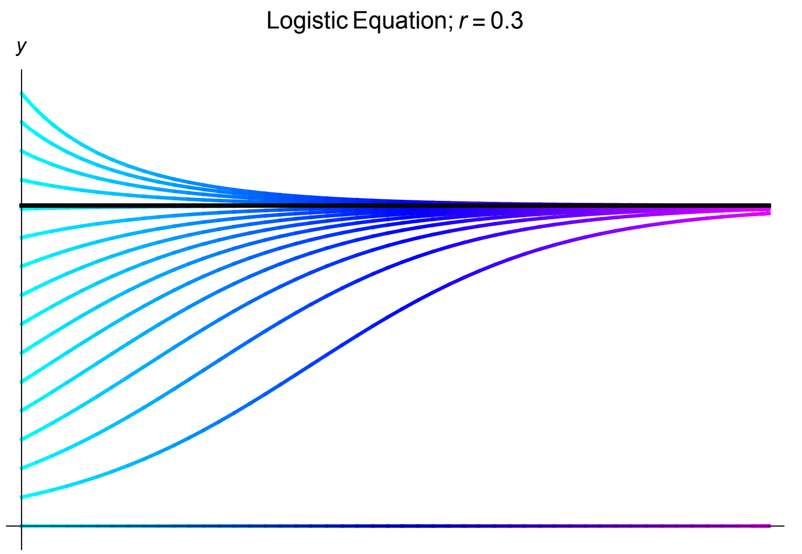
$$\Rightarrow y=\frac{K}{1+\frac{K-y_0}{y_0}e^{-rt}}=\frac{Ky_0}{y_0+(K-y_0)e^{-rt}}$$

$\Rightarrow$ Solution to the logistic equation:

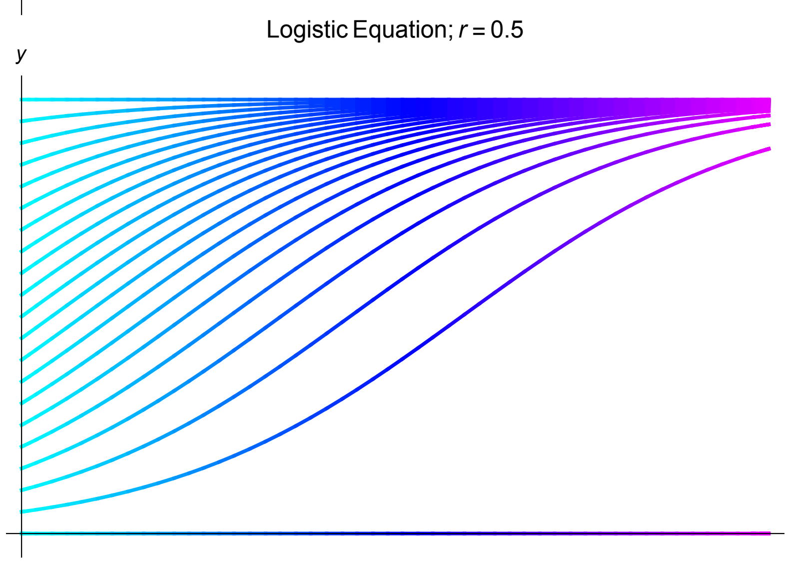
$$\boxed{y(t)=\frac{Ky_0}{y_0+(K-y_0)e^{-rt}}}$$



Exponential vs.Logistic Growth
y
t



Logistic Equation; r = 0.3
y
t



Logistic Equation; r = 0.5
y
t