

# A HYPERBOLIC DIFFEOMORPHISM WITH COUNTABLY MANY ERGODIC COMPONENTS NEAR IDENTITY

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ABSTRACT. We construct a smooth hyperbolic volume preserving diffeomorphism on a four dimensional compact Riemannian manifold which has countably many ergodic components and is arbitrarily close to the identity map.

## 0. INTRODUCTION

In 1978 Ya. Pesin proved that a hyperbolic volume preserving diffeomorphism  $f$  on a compact Riemannian manifold  $M$  admits at most countably many ergodic components (mod 0) by showing that almost every ergodic component has positive volume ([P]). Here hyperbolicity means nonzero Lyapunov exponents almost everywhere. However, an example of a system which has infinitely many ergodic components was unknown till 2000 when D. Doglopyat, Ya. Pesin and H. Hu constructed a diffeomorphism of the 3-dimensional torus that satisfies the properties stated above (see [DHP]).

The construction starts with a product  $F$  of a two dimensional toral automorphism  $A : \mathbf{T}^2 \rightarrow \mathbf{T}^2$  and the identity map on a circle,  $F = A \times id$ . The space is then partitioned into countably many subsets,  $M = \mathbf{T}^2 \times S^1 = \cup_{i=0}^{\infty} (\mathbf{T}^2 \times [\frac{1}{2^{i+1}}, \frac{1}{2^i}])$ . The diffeomorphism is then obtained by applying two small perturbations on each subset, one for ergodicity, and the other one for hyperbolicity. The latter one is a perturbation which removes the zero Lyapunov exponent along the interval direction. Since the original map  $F$  is a product of Anosov and identity, the resulting map will be away from the identity. In this paper we construct a system satisfying the properties in [DHP] in the proximity of the identity map. Our map is a perturbed geodesic flow on a surface of constant negative curvature instead of a toral automorphism. We will need three perturbations instead of two since there are two zero Lyapunov exponents to remove.

There are two difficulties in this construction. First, to obtain ergodicity we need accessibility. For this purpose we need to show that almost every point is accessible to a given central leaf along the interval direction. Since both strong stable and unstable leaves are one dimensional, the accessibility is easy to achieve in a three dimensional space as in [DHP] and extra work is necessary in our case

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because the space we work with is four dimensional. Second, the technique to remove the second zero Lyapunov exponent is more delicate. While we change the last Lyapunov exponent, we need to keep all other exponents nonzero at the same time. But the direction corresponding to the “small” nonzero exponent obtained by the earlier perturbation is not stable under perturbation.

Our results show that systems which are hyperbolic, volume preserving, and with countably many ergodic components exist in any neighborhood of the identity. However, the phenomenon may not be true for other properties of dynamical systems. For example, it is generally believed that there is no uniformly hyperbolic (Anosov) system in a neighborhood of the identity.

It is also interesting to compare our theorem with the results in [CS] and [X]. They show that on any manifold  $M$  of dimension at least two, there are open sets of volume preserving diffeomorphisms of  $M$ , all of which have positive measure sets on which all of the Lyapunov exponents are zero. In the terminology used in [BPSW], they prove that there are open sets of *neutral* volume preserving diffeomorphisms *away* from the identity, while we show that there are *(fully) nonuniformly hyperbolic* systems *near* identity.

## 1. STATEMENT OF RESULTS

We prove the following result.

**Theorem.** *There exists a four dimensional compact Riemannian manifold  $M$  such that for any  $\delta_0 > 0$ , we can find a  $C^\infty$  diffeomorphism  $f$  of  $M$  with the following properties:*

- (1)  $\|f - \text{id}\|_{C_1} \leq \delta_0$ ;
- (2)  $f$  preserves the Riemannian volume  $\mu$  on  $M$ ;
- (3)  $\mu$  is a hyperbolic measure;
- (4)  $f$  has countably many ergodic components which are open (mod 0).

## 2. CONSTRUCTION

Let  $g^t : M_0 \rightarrow M_0$  be a geodesic flow on a compact surface of a negative constant curvature.

Choose a closed orbit  $\mathcal{C}$ .

Take  $0 < \delta \leq \delta_0/2$  such that all points in  $\mathcal{C}$  are periodic points of  $g^\delta$ . Take  $G = g^\delta$ .

Let  $F = G \times \text{id}$  be the map from  $M = M_0 \times \mathbb{S}^1$  to itself. We will perturb  $F$  to obtain the desired map  $f$ .

Consider a countable collection of intervals  $\{I_n\}_{n=1}^\infty$  on the circle  $\mathbb{S}^1$ , where

$$I_{2n} = [(n+2)^{-1}, (n+1)^{-1}], \quad I_{2n-1} = [1 - (n+1)^{-1}, 1 - (n+2)^{-1}].$$

Clearly,  $\bigcup_{n=1}^\infty I_n = (0, 1)$  and  $\text{int } I_n$  are pairwise disjoint.

Choose  $\delta' > 0$  small enough. By Main Proposition below, for each  $n$  one can construct a  $C^\infty$  volume preserving ergodic hyperbolic diffeomorphism  $f_n : M_0 \times I \rightarrow M_0 \times I$  satisfying: 1)  $\|F - f_n\|_{C_1} \leq \delta' n^{-4}$ ; 2) for all  $0 \leq n < \infty$ ,  $D^k f_n|_{M_0 \times \{s\}} = D^k F|_{M_0 \times \{s\}}$  for  $s = 0$  or  $1$ .

Let  $L_n : I_n \rightarrow I$  be the affine map and  $\pi_n = (\text{id}, L_n) : M_0 \times I_n \rightarrow M_0 \times I$ . Clearly,  $\|\pi_n\| \leq 5n^2$  and  $\|\pi_n^{-1}\| \leq 1$ .

We define the map  $f$  by setting  $f|_{M_0 \times I_n} = \pi_n^{-1} f_n \pi_n$  for all  $n$  and then letting  $f|_{M_0 \times \{0\}} = F|_{M_0 \times \{0\}}$ . Note that

$$\|F|_{M_0 \times I_n} - \pi_n^{-1} f_n \pi_n\|_{C_1} \leq \|\pi_n^{-1} (F - f_n) \pi_n\|_{C_1} \leq \delta' n^{-4} \cdot 5n^2 = 5\delta' n^{-2} \leq \delta.$$

It follows that  $f$  is  $C^\infty$  on  $M$  and has the required properties.

### 3. MAIN PROPOSITION

The goal of this section is to prove the following statement.

**Main Proposition.** *Let  $S = G \times \text{id}$  be the diffeomorphism from  $N = M_0 \times I$  to itself. For any  $\delta > 0$ , there exists a map  $P$  such that:*

- (1)  $P$  is a  $C^\infty$  volume preserving diffeomorphism of  $N$ ;
- (2)  $\|S - P\|_{C_1} \leq \delta$ ;
- (3) for all  $0 \leq n < \infty$ ,  $D^n P|_{M_0 \times \{s\}} = D^n S|_{M_0 \times \{s\}}$  if  $s = 0$  and  $1$ ;
- (4)  $P$  is ergodic with respect to the Riemannian volume and has non-zero Lyapunov exponents almost everywhere.

Note that  $S$  is not ergodic, and has two zero Lyapunov exponents. We perturb  $S$  by three small perturbations  $h^{(i)} : \Omega_i \rightarrow \Omega_i$ ,  $i = 1, 2, 3$ , where  $\Omega_i \subset N$ , to get ergodicity and to remove zero Lyapunov exponents.

**Proof of Main Proposition:** Note that the tangent bundle of  $N$  can be written as a direct sum of four one-dimensional  $S$ -invariant subbundles:

$$TN = E^u(S) \oplus E^s(S) \oplus E^c(S) \oplus E^n(S),$$

where  $E^u(S)$ ,  $E^s(S)$  and  $E^c(S)$  are the unstable, stable and flow directions of the geodesic flow  $g^t : M_0 \rightarrow M_0$ , and  $E^n(S)$  is the tangent space of  $I$ . The corresponding Lyapunov exponents of  $S$  at  $w \in N$  are denoted by  $\lambda^u(w, S)$ ,  $\lambda^s(w, S)$ ,  $\lambda^c(w, S)$  and  $\lambda^n(w, S)$  respectively. It is clear that the Lyapunov exponents are constants at almost every point  $w \in N$ , though  $S$  is not ergodic. We simply denote them by  $\lambda^u(S)$  etc. Also, we know that  $\lambda^u(S) > 0$ ,  $\lambda^s(S) < 0$  and  $\lambda^c(S) = \lambda^n(S) = 0$ . We will often take local coordinate system  $w = (x, y, t, z)$  in  $N$  in such a way that

$$(3.1) \quad d\mu = dx dy dt dz, \quad \text{where } \frac{\partial}{\partial x} = E^u(S), \quad \frac{\partial}{\partial t} = E^c(S), \quad \frac{\partial}{\partial z} = E^n(S).$$

Fix  $\tau \in (0, 2/3)$ .

Take  $k_0 \in \mathbb{N}$  such that for  $\gamma \leq \delta$ ,  $\tau_2 = 1/4$ , some  $C > 0$ , and  $\theta = \pi/k_0$  such that the properties stated in Lemma 7.3 are satisfied.

Recall that  $\mathcal{C}$  is the closed orbit taken in Section 2. Take another closed orbit  $\mathcal{C}' \subset M_0$  of  $g^t$ .

Choose a set  $\Omega_0$  and constant  $\varepsilon_0$  according to Lemma 7.2 with  $\tau_1 = (1 - 1.3\tau)(k_0 + 1)^{-1}$ . Hence

$$(3.2) \quad \mu\left(\bigcup_{0 \leq i \leq k_0} G^i \Omega_0\right) \leq 1 - 1.3\tau.$$

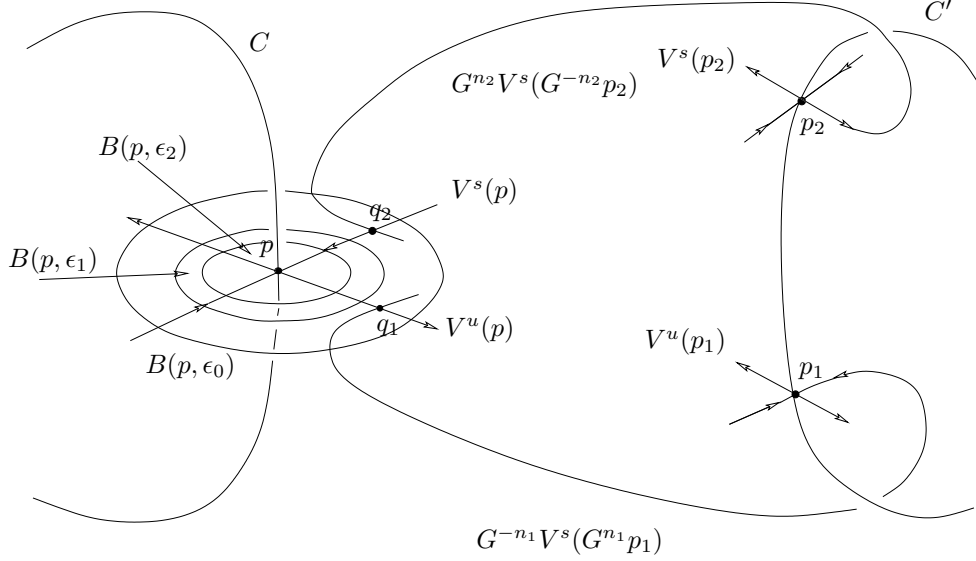
We also assume that  $\varepsilon_0$  is small enough such that  $\mathcal{C}$  and  $\mathcal{C}'$  are at least  $3\varepsilon_0$  separated.

*Construction of  $h^{(1)} : \Omega_1 \rightarrow \Omega_1$ .*

Fix  $p \in \mathcal{C}$ , a periodic point of  $G$  with period  $m \in \mathbb{N}$ .

We assume further that  $\varepsilon_0 > 0$  is small such that for any  $w \in N$ ,

$$(3.3) \quad \mu(B(w, \varepsilon_0)) < \frac{0.1\tau}{k_0 m}$$

FIGURE 1. Construction of  $h^{(1)}$ .

Here  $B(w, \varepsilon_0)$  denotes the ball in  $M_0$  of radius  $\varepsilon_0$  centered at  $w$ .

We assume that  $G^i(B(p, \varepsilon_0)) \cap B(p, \varepsilon_0) = \emptyset$  for  $i = 1, \dots, m-1$ .

For any  $z \in M_0$ , let  $V^u(z)$  and  $V^s(z)$  be the local unstable and stable one dimensional manifold at  $z$  for  $G$  of “size”  $\varepsilon_0$ .

Since both  $W^u(C')$  and  $W^s(C')$  are dense in  $M_0$ , we can choose  $p_1, p_2 \in C'$ , and the smallest integers  $n_1, n_2 > 0$  such that each intersection

$$G^{-n_1}V^s(G^{n_1}p_1) \cap V^u(p) \cap B(p, \varepsilon_0) \quad \text{and} \quad G^{-n_2}V^u(G^{-n_2}p_2) \cap V^s(p) \cap B(p, \varepsilon_0)$$

consists of a single point  $q_1$  and  $q_2$  respectively.

Take  $\varepsilon_1 \leq \min\{\delta, d(p, q_1)/2, d(p, q_2)/2\}$ . Take  $\ell \geq 2$  such that

$$(3.4) \quad G^{-\ell m}(q_1) \notin B(p, \varepsilon_1), \quad G^{-(\ell+1)m}(q_1) \in B(p, \varepsilon_1).$$

Then we take  $\varepsilon_2 \in (0, \varepsilon_1)$  such that  $G^{-(\ell+1)m}(q_1) \in B(p, \varepsilon_2)$ .

Let  $\Omega_1 = B(p, \varepsilon_0) \times I$ . Denote

$$(3.5) \quad \tilde{\Omega}_1 = \Omega_1 \bigcup (B(C, \varepsilon_3) \times I) \bigcup (\cup_{i=1}^{m-1} G^i(B(p, \varepsilon_0)) \times I) \bigcup \Omega'_1 \bigcup \Omega''_1,$$

where

$$\Omega'_1 = B(\cup_{i=0}^{\infty} G^{-n_1+i}V^s(G^{n_1}p_1), \varepsilon_3) \times I,$$

$$\Omega''_1 = B(\cup_{i=0}^{\infty} G^{-n_2-i}V^u(G^{-n_2}p_2), \varepsilon_3) \times I,$$

and  $B(\Omega, \varepsilon)$  is the  $\varepsilon$ -neighborhood of the set  $\Omega$  in  $M_0$  and  $\varepsilon_3$  is chosen such that

$$(3.6) \quad \mu\tilde{\Omega}_1 \leq 0.1\tau/k_0.$$

This is possible because of (3.3).

Take a coordinate system  $w = (x, y, t, z)$  in  $\Omega_1$  satisfying (3.1). Take the  $y$ -coordinate in such a way that  $\frac{\partial}{\partial y} = E^s(S)$  along the path  $V^s(p)$ .

Choose a  $C^\infty$  function  $\phi = \phi(r) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  which satisfies

- (1)  $\phi(r) = \phi_0$  if  $r \in [0, \varepsilon_2]$ , where  $\phi_0$  is a positive constants;
- (2)  $\phi(x) = 0$  if  $x \geq \varepsilon_1$ ;

- (3)  $\phi'(x) \leq 0$  for any  $x$ ;
- (4)  $\|\phi\|_{C^1} \leq \delta$ .

Choose a  $C^\infty$  function  $\psi = \psi(y) : \mathbb{R} \rightarrow \mathbb{R}^+$  which satisfies

- 5.  $\psi(x) = \psi_0$  if  $x \in (-\varepsilon_2, \varepsilon_2)$ , where  $\psi_0$  is a positive constant;
- 6.  $\psi(x) = 0$  if  $|x| \geq \varepsilon_1$ ;
- 7.  $\|\psi\|_{C^1} \leq \delta$ ;
- 8.  $\int_0^{\pm\varepsilon_1} \psi(s) ds = 0$ .

We also choose a  $C^\infty$  functions  $\xi : I \rightarrow \mathbb{R}^+$  satisfying:

- 9.  $\xi(s) > 0$  on  $(0, 1)$ ;
- 10.  $\xi^{(i)}(0) = \xi^{(i)}(1) = 0$  for  $i = 0, 1, 2, \dots$ ;
- 11.  $\|\xi\|_{C^1} \leq \delta$ .

Then we define the vector field  $X$  on  $\Omega_1$  by

$$X(x, y, t, z) = \left( -\phi(\sqrt{y^2 + t^2})\xi'(z) \int_0^x \psi(u) du, \quad 0, \quad 0, \quad \phi(\sqrt{y^2 + t^2})\xi(z)\psi(x) \right).$$

It is easy to check that  $X$  is a divergence free vector field supported in  $(-\varepsilon_1, \varepsilon_1) \times (-\varepsilon_1, \varepsilon_1) \times (-\varepsilon_1, \varepsilon_1) \times I \in \Omega_1$ . We define the map  $h^{(1)} = h_\beta^{(1)}$  on  $\Omega_1$  to be the time  $\beta$  map of the flow generated by  $X$  and we set  $h^{(1)} = \text{id}$  on the complement of  $\Omega_1$ . It is easy to see that  $h^{(1)}$  is a  $C^\infty$  volume preserving diffeomorphism. Also, we assume that  $\beta$  is small enough such that  $\|h^{(1)} - \text{id}\| \leq \delta$ .

Let  $R = R_\beta = h_\beta^{(1)} \circ S$  for some small  $\beta > 0$ . By Proposition 4.1, for any small  $\beta > 0$ ,  $R : N \rightarrow N$  is ergodic.

*Construction of  $h^{(2)} : \Omega_2 \rightarrow \Omega_2$ .*

Choose a point  $w^* = (w_0^*, z_0) \in N$ , where  $w_0^* \in M_0$  and  $z_0$  is the midpoint of  $I$ . Choose  $0 < \varepsilon_4 \leq \varepsilon_0$ . Let  $\Omega_2 = B(w^*, \varepsilon_4)$ , where  $B(w, \varepsilon)$  denotes the ball in  $N$  of radius  $\varepsilon$  centered at  $w \in N$ . We assume that  $w_0^*$  and  $\varepsilon_4$  is chosen in such a way that  $\Omega_2 \cap (\tilde{\Omega}_1 \cup (\Omega_0 \times I)) = \emptyset$  and  $S^{-i}\Omega_2 \cap (\Omega_2 \cup \Omega_1) = \emptyset$  for  $i = 1, \dots, N_0$ , where  $N_0$  is an integer large enough such that (B. 19) in [DHP] holds for this  $N_0$ . We also assume that  $\varepsilon_4$  is small enough such that

$$(3.7) \quad \mu(\Omega_2) \leq \frac{0.1\tau}{k_0}.$$

Let  $\rho(r) = (\varepsilon_4/\varepsilon_1)\phi(r\varepsilon_1/\varepsilon_4)$ . To define  $h^{(2)}$ , we take the cylindrical coordinate system  $(r, \theta, y, t)$  in  $\Omega_2$ , where  $x = r \cos \theta$ ,  $y = y$ ,  $t = t$  and  $z = r \sin \theta$ . Define  $h^{(2)} = h_\alpha^{(2)}$  on  $N$  by

$$(3.8) \quad h^{(2)}(r, \theta, y, t) = (r, \theta + \alpha\rho(\sqrt{y^2 + t^2})\rho(r), y, t).$$

Then we extend  $h^{(2)}$  to  $N$  by letting  $h^{(2)} = \text{id}$  on  $\Omega_2^c$ .

Let  $Q = Q_{\alpha\beta} = R_\beta \circ h_\alpha^{(2)} = h_\beta^{(1)} \circ S \circ h_\alpha^{(2)}$ .

We denote by  $\eta_{\alpha\beta}(w)$  the expanding rate of  $Q_{\alpha\beta}$  along its unstable direction  $E_w^u(Q_{\alpha\beta})$ . By Proposition 5.1, we have  $\int_N \log \eta_{\alpha\beta}(w) dw < \lambda^u(S)$ . Since  $\eta_{\alpha\beta}$  change smoothly with  $\beta$  we conclude that for some small  $\beta > 0$

$$\int_N \log \eta_{\alpha\beta}(w) dw < \lambda^u(S).$$

By Proposition 4.1,  $Q_{\alpha\beta}$  is ergodic. So we denote by  $\lambda^u(Q_{\alpha\beta})$  the largest Lyapunov exponent of  $Q_{\alpha\beta}$ . By the above inequality,  $\lambda^u(Q) \leq \lambda^u(S)$ .

Note that both  $Dh^{(1)}$  and  $Dh^{(2)}$  preserve  $E^{un}(S)$  bundle, and for any  $w \in N$ ,  $|\det Dh_w^{(1)}|_{E^{un}(S)}| = |\det Dh_w^{(2)}|_{E^{un}(S)}| = 1$ . So  $DQ$  preserves  $E^{un}(S)$  bundle,

and  $|\det DQ_w|_{E^{un}(S)}| = |\det DS_w|_{E^{un}(S)}|$ . Let  $\lambda^n(Q)$  denote the other Lyapunov exponent on  $E^{un}(S)$ . Then we have

$$\lambda^u(Q) + \lambda^n(Q) = \lambda^u(S) + \lambda^n(S).$$

Since  $\lambda^u(Q) < \lambda^u(S)$  and  $\lambda^n(S) = 0$ , we have  $\lambda^n(Q) > 0$ .

Note that the perturbations also preserve  $E^{ucn}(S)$  bundle. So we have

$$\lambda^u(Q) + \lambda^c(Q) + \lambda^n(Q) = \lambda^u(S) + \lambda^c(S) + \lambda^n(S)$$

and therefore  $\lambda^c(Q) = 0$ , where  $\lambda^c(Q)$  is the third Lyapunov exponent of  $Q$  on  $E^{ucn}(S)$ . Applying the same arguments, we also get  $\lambda^s(Q) = \lambda^s(S) < 0$ , though the stable bundle  $E^s(Q)$  may not be equal to  $E^s(S)$ .

*Construction of  $h^{(3)} : \Omega_3 \rightarrow \Omega_3$ .*

Denote  $\lambda = \lambda^n(Q)$ . We assume that  $\lambda$  is small in comparison with  $\lambda^u(Q)$ .

Denote

$$\Lambda' = \Lambda'(K) = \{w \in N : |\log |DQ_w^k|_{E^n(w,Q)}| - k\lambda| \leq 0.1k\lambda, \forall |k| \geq 0.5K\},$$

and  $\Lambda = \bigcap_{i=0}^{k_0} Q^{-i}\Lambda'$ . Note that  $\mu\Lambda' \rightarrow 1$  as  $K \rightarrow \infty$ . We assume that  $K$  satisfies

$$(3.9) \quad K\lambda \geq \max\{2k_0\lambda, 1.25 \log 2, -10k_0 \log(1 - \delta)\},$$

$$(3.10) \quad 0.001\tau^2\lambda + \mu\Lambda^c \log(1 - \delta) > 0,$$

$$(3.11) \quad \mu\Lambda^c \leq 0.1\tau,$$

where  $\Lambda^c$  is the complement of  $\Lambda(K)$  in  $N$ .

Note that if  $w \in \Lambda'$  then  $\|DQ_w^n(v)\| \geq e^{0.9n\lambda}\|v\|$  for  $n \geq 0.5K$ ,  $v \in E_w^{un}(Q)$ .

Denote

$$\Omega = \Lambda^c \bigcup \left( \bigcup_{i=0}^{k_0} Q^{-i}(\Omega_0 \cup \tilde{\Omega}_1 \cup \Omega_2) \right).$$

By (3.11), (3.2), (3.6) and (3.7),

$$(3.12) \quad \mu\Omega^c \geq 1 - 0.1\tau - (1 - 1.3\tau) - 0.1\tau - 0.1\tau = \tau.$$

Choose a set  $\Gamma' \in N$  such that  $Q^i\Gamma' \cap \Gamma' = \emptyset$ ,  $-K \leq i \leq 5\tau^{-1}K - 2K - 1$ ,  $i \neq 0$ . Here we assume that  $\tau^{-1}$  is an integer, otherwise we can use a smaller  $\tau$  instead. Denote

$$\bar{\Gamma}' = \bigcup_{i=-K}^{5\tau^{-1}K - 2K - 1} Q^i\Gamma'.$$

We also require that  $\mu\bar{\Gamma}'$  is close to 1 such that

$$(3.13) \quad (1 - 0.5\tau) \cdot \mu\bar{\Gamma}' \geq 1 - 0.6\tau.$$

The choice of such  $\Gamma'$  is possible because of the Rokhlin-Halmos Lemma.

We define

$$\Gamma_0 = \{Q^j w : w \in \Gamma', 0 \leq j \leq 4\tau^{-1}K - K - k_0, Q^j w \in \Omega^c, Q^i w \notin \Omega^c \text{ for } i \leq j\}.$$

In other words,  $\Gamma_0$  is the set of points from each trajectory  $\{Q^i w\}_{i=0}^{4\tau^{-1}K - K - k_0}$  that enter the set  $\Omega^c$  the first time.

Clearly  $Q^i\Gamma$ ,  $i = -K, \dots, K + k_0$ , are pairwise disjoint. Let  $\Gamma_i = Q^i\Gamma$  for  $i = -K, \dots, K + k_0$ ,  $\Gamma_{j,k} = \bigcup_{i=j}^k \Gamma_i$  for  $j \leq k$ , and in particular,  $\bar{\Gamma} = \Gamma_{-K, K+k_0}$ . Since  $\bar{\Gamma}$  is disjoint with  $\Omega$ , it is clear that  $\Gamma_i \cap (\Omega_0 \cup \tilde{\Omega}_1 \cup \Omega_2) = \emptyset$  for  $i = 1, \dots, k_0$ .

Since we are going to make perturbation  $h^{(3)}$  around the set  $\Gamma_{0,k_0-1}$ , this condition guarantees that the properties of  $Q$  we mentioned above still remain.

Approximate  $\Gamma = \Gamma_0$  by finitely many number of disjoint sets of the form

$$\Delta_{0j} = B^u(x_j, r'_j) \times B^s(y_j, r''_j) \times B^{cn}((t_j, z_j), r_j),$$

where  $w_i = (x_j, y_j, t_j, z_j) \in N$ ,  $r'_j, r''_j \geq r_j$  for  $j = 1, \dots, J$  and  $B^u, B^s$  and  $B^{cn}$  are the balls that correspond to the  $x, y$  and  $zt$ -coordinates. Denote  $\Delta_{ij} = Q^i \Delta_{0j}$ ,  $\Delta_i = \cup_{j=1}^J \Delta_{ij}$  for  $i = -K, \dots, K+k_0$ . We can pick  $\Delta_{0j}$  in such a way that  $\Delta_{ij} \cap \Delta_{kl} = \emptyset$  for  $(i, j) \neq (k, l)$ ,  $-K \leq i, k \leq K+k_0$ ,  $1 \leq j, l \leq J$  and  $\Delta_{ij} \cap (\Omega_0 \cup \tilde{\Omega}_1 \cup \Omega_2) = \emptyset$  for  $0 \leq i \leq k_0$ ,  $0 \leq j \leq J$ .

Also, denote  $\bar{\Delta} = \cup_{i=0}^{k_0-1} \Delta_i$ . Clearly,  $\Delta_i$  is an approximation of  $\Gamma_i$  for  $i = 1, \dots, k_0$ . We may assume that for each  $i = 0, \dots, k_0$ ,

$$(3.14) \quad \mu(\Gamma_i \Delta_i) \leq 0.05 \max\{\mu\Gamma_i, \mu\Delta_i\}.$$

Take  $\Omega_3 = \bar{\Delta}$ . On each  $\Delta_{ij}$  we apply Lemma 7.3 to get a map  $h = h_{ij}$  and a subset  $\Delta'_{ij} \subset \Delta_{ij}$  such that  $\|h_{ij} - \text{id}\| \leq \delta$ ,  $\mu\Delta'_{ij}/\mu\Delta_{ij} \geq 3/4$  and restricted to  $\Delta'_{ij}$ ,  $h$  is a rotation of angle  $\pi/2k_0$  along the  $cn$  plane. Note that  $DS|_{E^{cn}} = \text{id}$ , we can require that  $Q\Delta'_{ij} = \Delta'_{i+1,j}$  for  $j = 0, \dots, k_0 - 1$ .

Let  $\Delta'_i = \cup_{j=1}^J \Delta'_{ij}$ .

Let  $h^{(3)} = h_{ij}$  on each  $\Delta_{ij}$  and  $h^{(3)} = \text{id}$  otherwise. Take  $P = Q \circ h^{(3)}$ . By Proposition 4.1,  $P$  is ergodic. By Proposition 6.1, we have that for almost every  $w \in N$ , for all  $v \in E^{ucn}(w, S)$ ,  $\lambda(w, v, P) > 0$ . So  $P$  has three positive Lyapunov exponents. Since the foliation  $W^{ucn}(S)$  is preserved, we have  $E^{ucn}(w, S) = E^{ucn}(w, P)$ . This implies  $\lambda^s(P) = \lambda^s(S) < 0$ .  $\square$

#### 4. ERGODICITY

Recall that for a diffeomorphism  $f : M \rightarrow M$ , two points  $w_1, w_2 \in M$  are called *accessible (with respect to  $f$ )* if they can be joined by a piecewise differentiable piecewise nonsingular path which consists of segments tangent to either  $E^u(f)$  or  $E^s(f)$ . The diffeomorphism  $f$  is *essentially accessible* if almost any two points in  $M$  (with respect to the Riemannian volume) are accessible.

**Proposition 4.1.** *For any  $\alpha, \beta, \gamma > 0$  sufficiently small, the diffeomorphisms  $R = R_\alpha$ ,  $Q = Q_{\alpha\beta}$  and  $P = P_{\alpha\beta\gamma}$  of  $N$  are ergodic with respect to the Riemannian volume.*

**Proof:** By a result of Pugh and Shub, if a  $C^2$  diffeomorphism is partially hyperbolic, center bunched, dynamically coherent and essentially accessible, then the diffeomorphism is ergodic. (See [PS], Theorem A, also see [BPSW], Theorem 2.2.)

Clearly, all  $S, R, Q$  and  $P$  are partially hyperbolic and center bunched if  $\alpha, \beta$  and  $\gamma$  are small.

Note that  $S$  is dynamically coherent and its center foliation is plaque expansive. Since  $R, Q$  and  $P$  are  $C^1$  close to  $S$ , they are all dynamically coherent if  $\alpha, \beta$  and  $\gamma$  are small by a theorem of Pugh and Shub ([PS], Theorem 2.3).

By the lemma below,  $R, Q$  and  $P$  are essentially accessible. So they are ergodic.  $\square$

**Lemma 4.2.** *Any two points  $w, w' \in \text{int } N$  are accessible with respect  $R, Q$  or  $P$ . Therefore,  $R, Q$  or  $P$  are essentially accessible.*

**Proof:** Recall that  $p \in \mathcal{C}$  is chosen to construct  $h^{(1)}$ . Denote  $I_p = \{p\} \times (0, 1)$ , and  $\bar{I}_p = \{p\} \times [0, 1]$ . For  $T = R, Q$  or  $P$ , let  $\mathcal{A}_T$  be the set of points that are accessible to some point in  $\bar{I}_p$  with respect to  $T$ . Since  $\mathcal{A}_T$  is both open and closed in  $N$ , we get  $\mathcal{A}_T = N$ . Further, since  $\partial N$  is unperturbed by any of  $h^{(i)}$ ,  $i = 1, 2, 3$ , any point in  $\text{int } N$  is accessible to a point in  $I_p$ .

By the Sublemma below, we know that any two points in  $I_p$  are accessible. Since the accessibility property is symmetric and transitive, we get that any two points in  $\text{int } N$  are accessible.  $\square$

**Lemma 4.3.** *Let  $T = R, Q$  or  $P$ . For any  $s \in (0, 1)$ ,*

$$(4.1) \quad \mathcal{A}_T(p, s) \supset I_p,$$

where  $\mathcal{A}_T(p, s)$  is the set of points accessible to  $(p, s) \in N$  with respect to  $T$ .

**Proof:** In the arguments below we only use the sets  $\tilde{\Omega}_1$  defined in (3.5),  $\Omega_0$  chosen in Lemma 7.2, and the fact that the strong stable and unstable foliations are continuous with the maps. Since  $\Omega_2$  and  $\Omega_3$  are disjoint with  $\Omega_0$  and  $\tilde{\Omega}_1$ , and  $h^{(i)}$  are identity outside  $\Omega_i$  for  $i = 2, 3$ , the arguments work for all  $R, Q$  and  $P$ . So we drop the subscript  $T$  in  $\mathcal{A}_T(p, s)$  and simply write  $\mathcal{A}(p, s)$  instead.

We use the coordinate system  $(x, y, t, z)$  in  $\Omega_1$  described as we construct  $h^{(1)}$ . Denote  $h = h_\alpha^{(1)}$ . Since the map  $h$  preserves the leaf  $I_p$ , we have that

$$h(\bar{0}, z) = (h^1(\bar{0}, z), h^2(\bar{0}, z), h^3(\bar{0}, z), h^4(\bar{0}, z)) = (\bar{0}, h^4(\bar{0}, z))$$

for all  $z \in (0, 1)$ , where  $\bar{0} = (0, 0, 0)$ . It suffices to show that for every  $z \in (0, 1)$ ,

$$(4.2) \quad \mathcal{A}(p, z) \supset \{(p, z') : z' \in [(h_t^{-\ell})^4(p, z), z]\},$$

where  $\ell$  is chosen by (3.4). In fact, since accessibility is a transitive relation and  $h_t^{-n}(p, z) \rightarrow (p, 0)$  for any  $z \in (0, 1)$  as  $n \rightarrow \infty$ , (4.2) implies that  $\mathcal{A}(p, z) \supset \{(p, z') : z' \in (0, z]\}$ . Since this holds true for all  $z \in (0, 1)$  and accessibility is a reflective relation, we obtain (4.1).

Now we proceed with the proof of (4.2).

Recall that  $p_1, p_2 \in \mathcal{C}'$ ,  $q_1 \in G^{-n_1}V^s(p_1) \cap V^u(p)$  and  $q_2 \in G^{n_2}V^u(p_2) \cap V^s(p)$ . For  $z_0 = z$ , we choose  $z_i$ ,  $i = 1, \dots, 5$ , such that

$$\begin{aligned} (q_1, z_1) &\in V^u((p, z_0), T), & G^{-n_1}V^s((p_1, z_2), T) &\ni (q_1, z_1), \\ & & (p_2, z_3) &\text{ is accessible to } (p_1, z_2), \\ (q_2, z_4) &\in G^{n_2}V^u((p_2, z_3), T), & V^s((p, z_5), T) &\ni (q_2, z_4). \end{aligned}$$

This means that  $(p, z_5) \in \mathcal{A}(p, z_0)$ .

Let  $\pi : N = M_0 \times I \rightarrow M_0$  be a projection. Note that  $\pi V^s(w, T) = V^s(\pi w, G)$  if  $w \in \{p\} \times I$  or  $G^{-n_1}V^s((p_1, G) \times I)$ , and  $\pi V^u(w, T) = V^u(\pi w, G)$  if  $w \in \{p\} \times I$  or  $G^{n_2}V^u((p_2, G) \times I)$ . In other words,  $V^s(\pi w, T) \in V^s(w, G) \times I$  and  $V^u(\pi w, T) \in V^u(w, G) \times I$ . Hence we know that  $z_1, z_2, z_4, z_5$  are uniquely determined by  $z_0, z_1, z_3, z_4$  respectively. We will show that

- (1)  $z_1 = (h^{-\ell})^4(p, z_0)$ ;
- (2)  $z_2 = z_1$ ;
- (3)  $z_3$  can be chosen arbitrarily close to  $z_2$ ;
- (4)  $z_4 = z_3$ ;
- (5)  $z_5 \leq z_4$ .



If so, we can get

$$(4.3) \quad z_5 \leq (h_t^{-\ell})^4(p, z_0).$$

By continuity, we conclude that

$$\{(p, z') : z' \in [z_5, z_0]\} \subset \mathcal{A}(p, z_0)$$

and therefore (4.2) follows.

By the construction of  $h = h_\alpha^{(1)}$ , we know that for any  $q \in B(p, \varepsilon_2)$ ,

$$(4.4) \quad h^4(q, z) = h^4(p, z).$$

Recall that  $p$  is a periodic point of  $G$  with period  $m$ . We have

$$T^{-m}(p, z_0) = h^{-1}S^{-m}(p, z_0) = h^{-1}(G^{-m}p, z_0) = (p, (h^{-1})^4(p, z_0)),$$

and therefore for any  $k \geq 1$ ,

$$(4.5) \quad T^{-km}(p, z_0) = (p, (h^{-k})^4(p, z_0)).$$

On the other hand,  $G^{-km}q_1 \notin B(p, \varepsilon)$  if  $k \leq \ell$ . It follows that  $T^{-km}(q_1, z_1) = S^{-km}(q_1, z_1) = (G^{-km}q_1, z_1)$ . Hence, by (3.4),

$$\begin{aligned} T^{-(\ell+1)m}(q_1, z_1) &= h^{-1}S^{-(\ell+1)m}(q_1, z_1) \\ &= h^{-1}(G^{-(\ell+1)m}q_1, z_1) = (q^{(1)}, (h^{-1})^4(p, z_1)) \end{aligned}$$

for some  $q^{(1)} \in B(p, \varepsilon_2)$  and therefore, for any  $k > 0$ ,

$$(4.6) \quad T^{-(\ell+k)m}(q_1, z_1) = (q^{(k)}, (h^{-k})^4(p, z_1))$$

for some  $q^{(k)} \in B(p, \varepsilon_2)$ . Since  $(q_1, z_1) \in V^u((p, z_0), T)$ , by (4.5) and (4.6),

$$d(T^{-km}(p, z_0), T^{-km}(q_1, z_1)) = d((p, (h^{-k})^4(p, z_0)), (q^{(k)}, (h^{-k})^4(p, z_1))) \rightarrow 0$$

as  $k \rightarrow \infty$ , and the convergence is exponentially fast. So by taking the  $z$  component, we have that  $|(h^{-k})^4(p, z_0) - (h^{-k+\ell})^4(p, z_1)| \rightarrow 0$  converges exponentially fast as  $k \rightarrow \infty$ . On the other hand, if  $z' \neq z''$ , then  $|(h^{-k})^4(p, z') - (h^{-k})^4(p, z'')| \rightarrow 0$  with a subexponential rate because both  $(h^{-k})^4(p, z')$  and  $(h^{-k})^4(p, z'')$  converge to  $(p, 0)$ ,  $Dh = \text{id}$  at  $(p, 0)$ , and  $h$  is a  $C^\infty$  diffeomorphism. So we get  $z_1 = (h^{-\ell})^4(p, z_0)$ . This proves (1).

By using the same arguments, and the fact that  $h^4(0, y, 0, z) \geq z$  for any  $z \in (0, 1)$ , we can show that the  $z$ -coordinate of  $h$  is non-increasing from  $(p, z_5)$  to  $(q_2, z_4)$  along  $V^s((p, z_5), T)$ . That is,  $z_4 \geq z_5$ . This is (5).

Since the periodic orbit  $\mathcal{C}'$  and the sets  $\cup_{i=0}^{\infty} G^{-n_1+i}V^s(G^{n_1}(p_1), G) \times (0, 1)$  and  $\cup_{i=0}^{\infty} G^{n_2-i}V^u(G^{-n_2}(p_2), G) \times (0, 1)$  are unperturbed, we know that the  $z$ -coordinates are constant along the stable leaves  $T^{-n_1}V^s((G^{n_1}(p_1), z), T)$  and the unstable leaves  $T^{n_2}V^u((G^{-n_2}(p_2), z), T)$ . So we get  $z_1 = z_2$  and  $z_3 = z_4$ , which are (2) and (4).

Now we prove (3).

Denote by  $\mathcal{C}'(p_1p_2)$  the part of closed orbit  $\mathcal{C}'$  from  $p_1$  to  $p_2$ . By Lemma 7.2, for any small  $\varepsilon > 0$ , we can choose a closed orbit  $\mathcal{C}'_\varepsilon \subset \Omega_0$  such that a part of  $\mathcal{C}'_\varepsilon$  is in the  $\varepsilon$ -neighborhood of  $\mathcal{C}'(p_1p_2)$  in the sense that for any  $p' \in \mathcal{C}'(p_1p_2)$ , there is  $p'' \in \mathcal{C}'_\varepsilon$  with  $d(p', p'') \leq \varepsilon$ .

Consider the map  $H = H_{\mathcal{C}'\mathcal{C}'_\varepsilon}$  from  $\mathcal{C}'(p_1p_2)$  to  $\mathcal{C}'$  given in Section 7. Starting from  $p_1$ , we can apply the map consequently to get a sequence of points  $H^{(1)}(p_1), H^{(2)}(p_1), \dots$ , where  $H^{(i+1)}(p_1) = H(H^{(i)}(p_1))$ . Let  $k'$  be such a number that  $H^{(k')}(p_1) \in \mathcal{C}'(p_1p_2)$  and  $H^{(k'+1)}(p_1) \notin \mathcal{C}'(p_1p_2)$ . Clearly,  $p_1$  and  $p^{(k')}$  can be joint by  $4k'$  local stable and unstable manifolds of  $G$  at points on either  $\mathcal{C}'$  or  $\mathcal{C}'_\varepsilon$ . Recall that the neighborhoods of both  $\mathcal{C}' \times I$  and  $\mathcal{C}'_\varepsilon \times I$  are unperturbed, and

so are the  $4k'$  local stable and unstable manifolds. We know then that  $(p_1, z_2)$  is accessible to  $(H^{(k')}(p_1), z_2)$  with respect to  $T$ .

Note that  $H^{(k')}(p_1) \in \mathcal{C}'(p_1 p_2)$  and  $H^{(k'+1)}(p_1) \notin \mathcal{C}'(p_1 p_2)$ . By continuity we can find a closer orbit  $\mathcal{O}$  such that  $H_{\mathcal{C}'\mathcal{O}}(H^{(k')}(p_1)) = p_2$ . Hence there is  $z_3 \in (0, 1)$  such that  $(H^{(k')}(p_1), z_2)$  and  $(p_2, z_3)$  are accessible. We can make  $H^{(k')}(p_1)$  arbitrarily close to  $p_2$  by taking  $\varepsilon$  sufficiently small. Also note that the strong stable and unstable manifolds change continuously with respect to the diffeomorphism in the space of partially hyperbolic systems. If  $H^{(k')}(p_1)$  is sufficiently close to  $p_2$ , then the 4 strong stable and unstable leaves to construct  $H_{\mathcal{C}'\mathcal{O}}$  are sufficiently short. Hence,  $z_3$  can be made arbitrarily close to  $z_2$ .  $\square$

## 5. HYPERBOLICITY OF THE MAP $Q$

**Proposition 5.1.** *There exists  $\alpha_0 > 0$  such that for any  $\alpha \in (0, \alpha_0)$ ,*

$$\int_N \log \eta_{\alpha 0}(w) dw < \lambda^u(S).$$

**Proof:** Since  $Dh^{(2)}(E^{un}(S)) = E^{un}(S)$  for any  $w \in N$ , given  $\alpha > 0$ , there exists a unique number  $a_\alpha(w)$  such that the vector  $v_\alpha(w) = (1, 0, 0, a_\alpha(w)) \in E^u(Q_{\alpha 0})$ . Hence  $DQ_{\alpha 0}(w)v_\alpha(w) = (\eta_\alpha(w), 0, 0, \eta_\alpha a_\alpha(w))$  for some  $\eta_\alpha > 1$ . The expanding rate of  $DQ_{\alpha 0}$  along its unstable direction is

$$\eta_{\alpha 0}(w) = \eta_\alpha(w) \frac{\sqrt{1 + a_\alpha(Q_{\alpha 0}(w))^2}}{\sqrt{1 + a_\alpha(w)^2}}$$

Let  $L_\alpha = \int_N \log \eta_{\alpha 0}(w) dw = \int_N \log \eta_\alpha(w) dw$ . The second equality is true because the map  $Q_{\alpha 0}$  preserves the Riemannian volume and therefore  $\int_N \log(1 + a_\alpha(Q_{\alpha 0}w)^2) dw = \int_N \log(1 + a_\alpha(w)^2) dw$ . We have that

$$\begin{aligned} Dh_\alpha^{(2)}|_{E^{un}} &= \begin{pmatrix} A(w, \alpha) & B(w, \alpha) \\ C(w, \alpha) & D(w, \alpha) \end{pmatrix} \\ &= \begin{pmatrix} r_x \cos \sigma - r \sigma_x \sin \sigma & r_t \cos \sigma - r \sigma_t \sin \sigma \\ r_x \sin \sigma + r \sigma_x \cos \sigma & r_t \sin \sigma + r \sigma_t \cos \sigma \end{pmatrix} \end{aligned}$$

where  $r_x = \cos \theta$ ,  $r_t = \sin \theta$  and  $\sigma = \theta + \alpha \rho(\sqrt{y^2 + z^2}) \rho(r)$  and then  $\sigma_x = \frac{\sin \theta}{r} + \alpha \rho(\sqrt{y^2 + z^2}) \rho'_r(r) \cos \theta$  and  $\sigma_t = \frac{\cos \theta}{r} + \alpha \rho(\sqrt{y^2 + z^2}) \rho'_r(r) \sin \theta$ .

By the same arguments as in the proof of Proposition B.6 in [DHP], we can get that

$$\frac{dL_\tau}{d\tau} \Big|_{\tau=0} = 0, \quad \frac{d^2 L_\tau}{d\tau^2} \Big|_{\tau=0} < 0.$$

So we can choose  $\alpha_0 > 0$  so small that  $L_\alpha < \log \lambda^u(S)$  for any  $\alpha \in (0, \alpha_0)$ .  $\square$

## 6. HYPERBOLICITY OF THE MAP $P$

**Proposition 6.1.** *For almost every  $w \in N$ ,  $\chi(x, v, P) > 0 \forall v \in E^{ucn}(x, S)$ .*

**Proof:** Denote  $\Delta_0^* = \Delta'_0 \cap \Lambda$ . Then set

$$U_1 = P^{-K} \Delta_0^*, \quad U_2 = \Delta_0 \setminus \Delta_0^*, \quad U_3 = \Delta_{k_0} \setminus P^{k_0} \Delta_0^*, \quad U_4 = P^{-K} (\Delta_0 \setminus \Delta_0^*).$$

Clearly,  $\{U_i, i = 1, 2, 3, 4\}$  are pairwise disjoint. Let  $U = U_1 \cup U_2 \cup U_3 \cup U_4$ . Note that  $\Delta_0^* \supset \Gamma \cap \Delta'_0$ . Let  $\bar{P} = P^\tau : U \rightarrow U$  be the first return map of  $P$ , where  $\tau = \tau(w)$  is the first return time of  $w \in U$ .

In the proof below, for any  $w \in U$ , we always assume  $v \in E^{ucn}(w)$ .

If  $w \in U_1$ , then  $\tau(w) \geq 2K + k_0$ . By Lemma 6.2 and (3.9),

$$(6.1) \quad \log \|D\bar{P}_w(v)\| \geq 0.9K\lambda - 0.5 \log 2 + \log \|v\| \geq 0.5K\lambda + \log \|v\|.$$

By (3.14),

$$(6.2) \quad \begin{aligned} \mu U_1 &= \mu(\Delta_0^*) \geq \mu(\Gamma \cap \Delta'_0) \geq \mu(\Delta'_0) - \mu(\Gamma \Delta \Delta'_0) \\ &\geq \frac{3}{4}\mu\Delta_0 - \mu(\Gamma_0 \Delta \Delta_0) \geq (0.75 - 0.05)\mu\Delta_0 = 0.7\mu\Delta_0. \end{aligned}$$

Note that  $\|DP - \text{id}\| \leq \delta$ . So if  $w \in U_2$ , then  $\bar{P} = P^{k_0}$  and

$$(6.3) \quad \log \|D\bar{P}_w(v)\| \geq k_0 \log(1 - \delta) + \log \|v\|.$$

Also,

$$(6.4) \quad \begin{aligned} \mu U_2 &= \mu(\Delta_0 \setminus \Delta_0^*) \leq \mu(\Delta_0 \setminus \Delta'_0) + \mu(\Delta'_0 \setminus \Delta_0^*) \leq \frac{1}{4}\mu(\Delta_0) + \mu(\Delta'_0 \setminus (\Gamma \cap \Delta'_0)) \\ &\leq \frac{1}{4}\mu\Delta_0 + \mu(\Gamma_0 \Delta \Delta_0) \leq (0.25 + 0.05)\mu\Delta_0 = 0.3\mu\Delta_0. \end{aligned}$$

Consider the case that  $w \in U_3$ . The construction of  $\{\Gamma_i\}$  and  $\{\Delta_i\}$  implies that  $\tau(w) \geq K$ . Also, observe that if  $P^i w \in \Omega_3$  for some  $i > 0$ , then  $P^i w \in \Delta_j$  for some  $0 \leq j \leq k_0 - 1$ . That is,  $P^{i-j} w \in \Delta_0$ . Hence we have either  $P^{i-j} w \in U_2 \subset U$  or  $P^{i-j-K} w \in U_1 \subset U$ , which implies that  $\tau(w) \geq i$ , and therefore the piece of orbit  $\{w, \dots, P^{\tau(w)-1} w\}$  does not intersect  $\Omega_3$ . So we have  $\bar{P}_w(v) = Q_w^{\tau(w)}(v)$  because  $P = Q$  on  $N \setminus \Omega_3$ .

Let  $\tau'(w)$  be the smallest positive integer such that  $P^{\tau'(w)} w \in \Lambda$  for some  $0 \leq \tau'(w) \leq \tau(w)$ , and let  $\tau'(w) = \tau(w)$  if there is no such integer. Denote

$$U'_3 = U_3 \cap \{w : \tau(w) - \tau'(w) \geq 0.5K\}, \quad U''_3 = U_3 \cap \{w : \tau(w) - \tau'(w) < 0.5K\},$$

and

$$\hat{U}'_3 = \{P^i w : w \in U'_3, 0 \leq i < \tau'(w)\}, \quad \hat{U}''_3 = \{P^i w : w \in U''_3, 0 \leq i < \tau'(w)\}.$$

Note that if  $n \geq 0.5K$  and  $w \in \Lambda$ , then  $\|DQ_w^n(v)\| \geq \|v\|$  for any  $v \in E^{ucn}(w)$ . Also note that  $P = Q$  on  $N \setminus \Omega_3$ .

If  $w \in U'_3$ , then

$$\|D\bar{P}_w(v)\| = \|DP_w^{\tau(w)}(v)\| = \|DQ_{P^{\tau'(w)}(w)}^{\tau(w)-\tau'(w)}(DP_w^{\tau'(w)}(v))\| \geq \|DP_w^{\tau'(w)}(v)\|.$$

Hence,

$$(6.5) \quad \log \|D\bar{P}_w(v)\| \geq \log \|P_w^{\tau'(w)}(v)\| \geq \sum_{i=0}^{\tau'(w)-1} \chi_{\hat{U}'_3}(P^i w) \log(1 - \delta) + \log \|v\|,$$

where  $\chi_\Omega(\cdot)$  is the characteristic function of the set  $\Omega$ .

If  $w \in U''_3$ , we denote

$$\tilde{U}''_3 = \{P^i w : w \in U''_3, 0 \leq i < \tau(w)\}.$$

Clearly, we have

$$(6.6) \quad \log \|D\bar{P}_w(v)\| \geq \sum_{i=0}^{\tau(w)-1} \chi_{\tilde{U}''_3}(P^i w) \log(1 - \delta) + \log \|v\|.$$

Lastly, we consider the case that  $w \in U_4$ . We have  $\tau(w) = K$  and  $\bar{P}(w) \in U_2$ . We define  $\tau''$  and  $U'_4, U''_4, \hat{U}'_4, \hat{U}''_4$  and  $\tilde{U}''_4$  in the same way as in the previous case, and get similar inequalities. That is, if  $w \in U'_4$ , then

$$(6.7) \quad \log \|D\bar{P}_w(v)\| \geq \sum_{i=0}^{\tau(w)-1} \chi_{\hat{U}'_4}(P^i w) \log(1 - \delta) + \log \|v\|,$$

and if  $w \in U''_4$ , then

$$(6.8) \quad \log \|D\bar{P}_w(v)\| \geq \sum_{i=0}^{\tau(w)-1} \chi_{\hat{U}''_4}(P^i w) \log(1 - \delta) + \log \|v\|,$$

Clearly,  $\hat{U}'_3, \hat{U}''_3, \hat{U}'_4, \hat{U}''_4 \in \Lambda^c$ . So

$$(6.9) \quad \mu(\hat{U}'_3 \cup \hat{U}''_3 \cup \hat{U}'_4 \cup \hat{U}''_4) \leq \mu\Lambda^c.$$

By the fact that  $\tau'', \tau' \geq 0.5K$  on  $\tilde{U}''_3$  and  $\tilde{U}''_4$  respectively, we have

$$(6.10) \quad \mu(\tilde{U}''_3 \cup \tilde{U}''_4) \leq 2\mu(\hat{U}''_3 \cup \hat{U}''_4) \leq 2\mu\Lambda^c$$

Now we estimate the Lyapunov exponent of  $v$  at a typical point  $w$ . We may assume  $w \in U$ . Let  $n_i = \tau_i(w)$  be the  $i$ th return time of  $w$ . Then we have

$$\lambda(v, w, P) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \frac{\|DP_w^n(v)\|}{\|v\|} = \lim_{i \rightarrow \infty} \frac{1}{n_i} \log \frac{\|DP_w^{n_i}(v)\|}{\|v\|}$$

Using the fact that the frequency of the orbit visiting a set  $U$  is equal to  $\mu U$ , and noticing (6.1)-(6.10), we have

$$\begin{aligned} \lim_{j \rightarrow \infty} \frac{1}{n_j} \log \frac{\|DP_w^{n_j} v\|}{\|v\|} &= \lim_{j \rightarrow \infty} \frac{1}{n_j} \sum_{i=0}^{j-1} \log \frac{\|D\bar{P}_{P^i w}(D\bar{P}_w^i(v))\|}{\|D\bar{P}_w^i(v)\|} \\ &\geq \mu U_1 \cdot 0.5K\lambda + \mu U_2 \cdot k_0 \log(1 - \delta) + (\mu \hat{U}'_3 + \mu \hat{U}'_4 + \mu \hat{U}''_3 + \mu \hat{U}''_4) \log(1 - \delta) \\ &\geq 0.7\mu\Delta_0 \cdot 0.5K\lambda + 0.3\mu\Delta_0 \cdot k_0 \log(1 - \delta) + 3\mu\Lambda^c \cdot \log(1 - \delta) \end{aligned}$$

Using (3.10) and (3.11) we conclude that the right side of the above inequality is greater than

$$0.33\mu\Delta_0 K\lambda - 0.003\tau^2\lambda.$$

Since  $\mu\Delta_0 \geq \mu\Gamma - \mu(\Gamma_0\Delta\Delta_0) \geq 0.08\tau^2 K^{-1} - 0.05(0.08\tau^2 K^{-1})$  by Lemma 6.3 and (3.14), we conclude finally that  $\lambda(v, w, P) > 0$ .  $\square$

**Lemma 6.2.** *Let  $w \in P^{-K}(\Delta'_0 \cap \Lambda)$ . Then for any  $v \in E^{un}(w, S)$ ,*

$$\|D\bar{P}_w(v)\| \geq \frac{\sqrt{2}}{2} \|v\| e^{0.9K\lambda}.$$

**Proof:** Note that on  $\Gamma_{-K, -1}$ ,  $h^{(3)} = \text{id}$ , hence  $P^K(w) = Q^K(w)$ . Also, since both  $Dh^{(1)}$  and  $Dh^{(2)}$  preserve the subbundle  $E^{un}(S)$ , we have  $E^{un}(w, Q) = E^{un}(w, S)$ .

Write  $v = v^{un} + v^c$ , where  $v^{un} \in E^{un}(w, Q)$  and  $v^c \in E^c(w, Q)$ .

We assume first  $\|v^c\| \leq \frac{\sqrt{2}}{2} \|v\|$ . Hence  $\|v^{un}\| \geq \frac{\sqrt{2}}{2} \|v\|$ . Since  $DQ^K(v^{un}) \in E^{un}(Q^K w, Q)$  and  $Q^K w \in \Lambda$ , we have

$$\|v^{un}\| = \|DQ^{-K}(DQ^K v^{un})\| \leq \|DQ^K v^{un}\| e^{-0.9K\lambda}.$$

Hence,

$$\|DP^K v\| = \|DQ^K v\| \geq \|DQ^K v^{un}\| \geq \|v^{un}\| e^{0.9K\lambda} \geq \frac{\sqrt{2}}{2} \|v\| e^{0.9K\lambda(Q)}.$$

Note that at the points  $P^K w, \dots, P^{K+k_0-1} w$ , the map  $Dh^{(3)}$  is a rotation, and  $DQ|_{E^{ucn}(P^i w)} = DS|_{E^{ucn}(P^i w)}$  is noncontracting for  $i = K, \dots, K+k_0-1$ . So  $DP^{k_0}|_{E^{ucn}(P^K w)}$  is noncontracting. Further, since  $\{P^i w\}_{i=K+K_0}^\tau \cap \Omega_3 = \emptyset$  and  $P^{K+k_0} w \in \Lambda'$ , we have that  $DP^{\tau-(K+k_0)}|_{E^{un}(P^{K+k_0} w)} = DQ^{\tau-(K+k_0)}|_{E^{un}(P^{K+k_0} w)}$  is expanding, and  $DP^{\tau-(K+k_0)}|_{E^{ucn}(P^{K+k_0} w)}$  is noncontracting. So we have

$$\begin{aligned} \|D\bar{P}(v)\| &= \|DP_{P^{K+k_0} w}^{\tau-(K+k_0)}(DP_w^{K+k_0}(v))\| = \|DQ_{P^{K+k_0} w}^{\tau-(K+k_0)}(DP_w^{K+k_0}(v))\| \\ &\geq \|DP_w^{K+k_0}(v)\| \geq \|DP_w^K(v)\| = \|DQ_w^K(v)\| \geq \frac{\sqrt{2}}{2} \|v\| e^{0.9K\lambda}. \end{aligned}$$

Now we consider the case that  $\|v^c\| \geq \frac{\sqrt{2}}{2} \|v\|$ .

Note that  $DQ^K(v^c) \in E^c(Q^K w, Q)$ . By the construction of  $h^{(3)}$ , we see that  $DP_{Q^K w}^{k_0}$  rotate the vector in  $E^{cn}(Q^K w, S)$  by  $\pi/2$ . It means that  $DP^{K+k_0}(v^c) = DP^{k_0}(DQ^K(v^c)) \in E^{un}(P^{K+k_0} w, Q)$ . Using the fact that  $P^{K+k_0} w \in \Lambda$ , we have

$$\begin{aligned} \|D\bar{P}(v^c)\| &= \|DP_{P^{K+k_0} w}^{\tau-(K+k_0)}(P_w^{K+k_0}(v^c))\| \geq \|DP^K(DP^{K+k_0}(v^c))\| \\ &\geq \|DP^{K+k_0}(v^c)\| e^{0.9K\lambda} \geq \|v^c\| e^{0.9K\lambda} \geq \frac{\sqrt{2}}{2} \|v\| e^{0.9K\lambda}. \end{aligned}$$

This is the result.  $\square$

**Lemma 6.3.**  $\mu\Gamma \geq 0.08\tau^2 K^{-1}$ .

**Proof:** Let

$$\hat{\Gamma}' = \bigcup_{i=0}^{5\tau^{-1}K-2K-k_0} Q^i \Gamma'.$$

Since  $\tau^{-1} \geq 2$  and  $K \geq 2k_0$ , we have

$$\frac{\mu\hat{\Gamma}'}{\mu\bar{\Gamma}'} = \frac{5\tau^{-1}K - 2K - k_0 + 1}{5\tau^{-1}K - 2K - 1 + K + 1} = 1 - \frac{K - k_0 + 1}{5\tau^{-1}K - K} \geq 1 - 0.5\tau.$$

By (3.13),  $\mu\hat{\Gamma}' = (1 - 0.5\tau)\mu\bar{\Gamma}' \geq 1 - 0.6\tau$ . Then by (3.12),

$$\mu(\hat{\Gamma}' \setminus \Omega) \geq 0.4\tau.$$

For  $w \in \Gamma'$ , we denote  $O(w) = \{Q^i w : i = 0, \dots, (5\tau^{-1} - 2)K - k_0\}$ , the piece of orbit that start at  $w$  from time 0 to  $(5\tau^{-1} - 2)K - k_0$ . Let

$$\Gamma'_a = \{O(w) : w \in \Gamma', O(w) \cap \Omega^c \neq \emptyset\}, \quad \Gamma'_b = \{O(w) : w \in \Gamma', O(w) \cap \Omega^c = \emptyset\}.$$

Clearly,  $\{\Gamma'_a, \Gamma'_b\}$  forms a partition of  $\hat{\Gamma}'$ , and  $\Gamma'_b \subset \Omega$  and therefore by (3.12),

$$\mu\Gamma'_a = \mu\hat{\Gamma}' - \mu\Gamma'_b \geq \mu\hat{\Gamma}' - \mu\Omega \geq (1 - 0.6\tau) - (1 - \tau) = 0.4\tau.$$

Note that  $\Gamma$  consists of exactly one point from each orbit  $O(w)$  in  $\Gamma_a$ . We get

$$\mu\Gamma \geq \frac{\mu\Gamma_a}{(5\tau^{-1} - 2)K - k_0 + 1} \geq \frac{0.4\tau}{(5\tau^{-1} - 2)K} \geq \frac{0.4\tau}{5\tau^{-1}K} = 0.08\tau^2 K^{-1}.$$

This is the result.  $\square$

## 7. PROPERTIES OF GEODESIC FLOWS

Let  $g^t : M_0 \rightarrow M_0$  be the geodesic flow on a compact surface of a negative constant curvature. We list some properties of the flow here.

- 1)  $d(g^t x, x) \leq |t|$  for any  $t \in \mathbb{R}$  and  $x \in M_0$ ;
- 2)  $g^t$  is a uniformly hyperbolic flow, that is, there is a decomposition of the tangent bundle into

$$TM_0 = E^u \oplus E^s \oplus E^c$$

and a constant  $\tilde{\eta} > 1$  such that for any  $z \in M_0$ ,

$$\begin{aligned} |Dg_z^t(v)| &\geq \tilde{\eta}^t |v| & v \in E_z^u, \\ |Dg_z^{-t}(v)| &\geq \tilde{\eta}^{-t} |v| & v \in E_z^s, \end{aligned}$$

and  $E^c$  is the one dimensional bundle tangent to the flow.

- 3) The closed orbits are dense in  $M_0$ . Moreover, for any closed orbit  $\mathcal{C}$ , both  $W^u(\mathcal{C}) = \cup_{z \in \mathcal{C}} W^u(z)$  and  $W^s(\mathcal{C}) = \cup_{z \in \mathcal{C}} W^s(z)$  are dense in  $M_0$ .

- 4)  $g^t$  preserves the Riemannian volume and for any  $t \neq 0$ ,  $g^t$  is ergodic with respect to the volume,

- 5)  $g^t$  has the accessibility property. That is, any two points can be joint by a piecewise differentiable piecewise nonsingular path which consists of segments tangent to either  $E^u$  or  $E^s$ .

- 6)  $g^t$  is topologically conjugate to a symbolic flow that is a suspension of a subshift finite type with a continuous roof function. More precisely, there is a symbolic space  $\Sigma_A$ , two sided left shift  $\sigma_A : \Sigma_A \rightarrow \Sigma_A$ , a continuous function  $\iota : \Sigma_A \rightarrow \mathbb{R}^+$  and a finite to one map  $\pi : \Sigma_A^t \rightarrow M_0$  such that

$$\pi \circ g^t = \bar{\sigma}_A^t \circ \pi,$$

where

$$\begin{aligned} \Sigma_A^t &= \{(\underline{w}, t) \in \Sigma_A \times \mathbb{R}\} / \{(\underline{w}, \iota(\underline{w})) = (\sigma_A(\underline{w}), 0)\}, \\ \bar{\sigma}_A^t(\underline{w}, s) &= (\underline{w}, s + t). \end{aligned}$$

Moreover,  $p \in M_0$  is periodic under  $g^t$  if and only if  $\pi^{-1}(p)$  is periodic under  $\bar{\sigma}_A^t$ . Also,  $\Sigma_A$  can be chosen in such a way that the size of  $\pi R_\alpha$  can be arbitrarily small, where  $R_\alpha = \{w = \cdots w_{-1} w_0 w_1 \cdots : w_0 = a\}$  is a cylinder in  $\Sigma_A$ . (See [B] and [BR] for more details.)

**Definition 7.1.** Let  $\mathcal{C}_\varepsilon$  and  $\mathcal{C}'$  be two orbits,  $p \in \mathcal{C}'$ ,  $x \in \mathcal{C}_\varepsilon$  and  $d(p, x)$  is small. Define the holonomy  $H_{\mathcal{C}', \mathcal{C}_\varepsilon} : B_{\mathcal{C}'}(p, \delta) \rightarrow \mathcal{C}'$  where  $B_{\mathcal{C}'}(p, \delta)$  is a  $\delta$ -neighborhood of point  $p$  in  $\mathcal{C}'$ .  $H_{\mathcal{C}', \mathcal{C}_\varepsilon}(x_0)$  is constructed the following way (see Figure 2):

- (1)  $x_1 = W^s(x_0) \cap W^{uc}(x)$ ;
- (2)  $x_2 = W^u(x_1) \cap W^{cs}(x) \subset W^u(x_1) \cap \mathcal{C}_\varepsilon$ ;
- (3)  $x_3 = W^s(x_2) \cap W^{cu}(x_0)$ ;
- (4)  $x_4 = W^u(x_3) \cap W^{cs}(x_0) \subset W^u(x_3) \cap \mathcal{C}'$ ;
- (5)  $H_{\mathcal{C}', \mathcal{C}_\varepsilon}(x_0) = x_4$ .

This holonomy map was used in [NT]. Note that in our case for any point  $x_0 \in \mathcal{C}'$ ,  $H_{\mathcal{C}', \mathcal{C}_\varepsilon}(x_0) \neq x_0$  because  $g$  is a geodesic flow on a compact surface of a negative curvature.

**Lemma 7.2.** Let  $\mathcal{C}'$  be a given closed orbits of  $g^t$ . For any  $\tau_1 \in (0, 1)$ , there is an open subset  $\Omega_0 \subset M_0$  containing  $\mathcal{C}'$  with  $\mu\Omega_0 \leq \tau_1$  and there exists a constant  $\varepsilon_0 > 0$  such that for any  $\varepsilon \in (0, \varepsilon_0)$ , there is a closed orbit  $\mathcal{C}_\varepsilon$  with  $d(\mathcal{C}', \mathcal{C}_\varepsilon) < \varepsilon$  and  $B(\mathcal{C}_\varepsilon, \varepsilon) \subset \Omega_0$ , where  $B(\mathcal{C}_\varepsilon, \varepsilon)$  denotes the  $\varepsilon$  neighborhood of  $\mathcal{C}_\varepsilon$  in  $M_0$ .

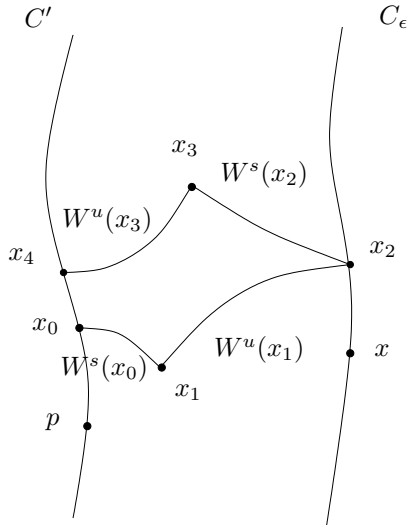


FIGURE 2. Holonomy on a closed orbit.

**Proof:** Let  $\sigma_A : \Sigma_A \rightarrow \Sigma_A$  be a symbolic flow that is conjugate to  $g^t : M_0 \rightarrow M_0$  with the conjugacy  $\pi : \Sigma_A \rightarrow M_0$ .

For a word  $W = s_i \cdots s_j$ ,  $i \leq j$ , we denote by  $\mathcal{C}(W)$  the periodic orbit of  $\sigma_A$ , and by  $\mathcal{R}(W)$  the cylinder determined by  $W$ . Abusing notations we denote by  $\pi\mathcal{C}(W)$  the corresponding closed orbit of  $g^t$  and by  $\pi\mathcal{R}(W)$  the corresponding set in  $M_0$ . More precisely, the latter means the set

$$\{(\pi\underline{w}, t) \in M_0 : \underline{w} \in \mathcal{R}(W), 0 \leq t \in \iota(\underline{w})\}.$$

Let  $W'$  be the word such that  $\pi\mathcal{C}(W') = \mathcal{C}'$  and  $W_a$  be a subword. Take another word  $W^*$  that generates a periodic orbit of  $\sigma_A$  and contains the same subword  $W_a$ . Such word  $W^*$  exists since the periodic orbits of  $\sigma_A$  are dense. We may assume that  $W' = W_a W_b$  and  $W^* = W_a W_c$  for some word  $W_b$  and  $W_c$ . It implies that one of the words  $W'$  and  $W^*$  can be followed by another.

Note that the maximal volume of all sets of the form  $\pi\mathcal{R}(W(n))$  converges to 0 exponentially fast, where  $W(n)$  is a  $(2n+1)$ -word of the form  $s_{-n} \cdots s_n$ . For any  $\tau_1 > 0$ , we can take  $n > 0$  such that  $\mu(\pi\mathcal{R}(W(n))) \leq \tau_1/(4n+2)$  for any  $(2n+1)$ -word  $W(n)$ . We may assume  $n > |W'|, |W^*|$ , where  $|W|$  denotes the length of  $W$ . Note that for any integer  $k \geq n/|W'|$ , there are at most  $4n+2$  different  $(2n+1)$ -words of the form  $W(n)$  in the orbit of  $\underline{w} = \mathcal{C}((W')^k W^*)$ , where  $W^k$  is the word consists  $k$  consecutive  $W$ . Let  $W_1, \dots, W_j$ ,  $j \leq 4n+2$ , denote these words.

Set

$$\hat{\Omega}_0 = \bigcup_{i=1}^j \pi\mathcal{R}(W_i).$$

Clearly,  $\mu\hat{\Omega}_0 \leq \tau_1/2$ . Choose  $\varepsilon_0 > 0$  such that  $\mu B(\hat{\Omega}_0, \varepsilon_0) \leq \tau_1$ . Then we set  $\Omega_0 = B(\hat{\Omega}_0, \varepsilon_0)$ .

Let  $\varepsilon \in (0, \varepsilon_0)$ . We take a word of the form  $(W')^k W^*$  for some large  $k$ , and then take  $\mathcal{C}_\varepsilon = \pi(\mathcal{C}((W')^k W^*))$ . Clearly,  $\mathcal{C}_\varepsilon \subset \hat{\Omega}_0$ , and therefore  $B(\mathcal{C}_\varepsilon, \varepsilon) \subset \Omega_0$  by the choice of  $\Omega_0$ . Also if  $k$  is large enough, then the distance between  $\mathcal{C}_\varepsilon$  and  $\mathcal{C}'$  can be made arbitrarily small. Hence we have  $\mathcal{C}' \subset B(\mathcal{C}_\varepsilon, \varepsilon)$ .  $\square$

**Lemma 7.3.** *For any  $\gamma > 0$ ,  $\tau_2 \in (0, 1)$  and  $C > 0$ , we can find a constant  $\theta_0 > 0$  such that for any number  $\theta \in [0, \theta_0]$ , any set of the form  $\Delta = \Delta_{ss's''} = B^u(x, s') \times B^s(y, s'') \times B^{cn}((t, z), s)$ , where  $s', s'' \geq s$  and  $s', s'' \leq C$ , there exists a set  $\Delta^{(0)}$  and a map  $h : N \rightarrow N$  with the following properties:*

- (a)  $h = T_\theta$  on  $\Delta^{(0)}$ , and  $h = \text{id}$  on  $\Delta^c$ ;
- (b)  $\mu\Delta^{(0)}/\mu\Delta \geq \tau_2$ ;
- (c)  $\|h - \text{id}\| \leq \gamma$ ,

where  $T_\theta$  is a rotation given by

$$T_\theta(x, y, t, z) = (x, y, t \cos \theta - z \sin \theta, t \sin \theta + z \cos \theta).$$

**Proof:** Take  $\kappa > 0$  such that  $\mu\Delta_{1-\kappa, 1-\kappa, 1-\kappa}/\mu\Delta_{111} \geq \tau_2$ . Hence, for any  $r > 0$ ,  $r', r'' > r$ , we have  $\mu\Delta_{r-\kappa r, r'-\kappa r, r''-\kappa r}/\mu\Delta_{r'r''} \geq \tau_2$ , since  $r'/(r' - \kappa r)$  and  $r''/(r'' - \kappa r)$  are increasing.

Take a family of  $C^\infty$  functions  $\zeta_r = \zeta_r(s) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ , for  $r \geq 1$  such that

- 1  $\zeta_1(s) = 1$  if  $s \in [0, 1 - \kappa]$  and  $\zeta_1(s) = 0$  if  $s \geq 1$ ;
- 2  $\zeta_r(s) = 1$  if  $s \in [0, r - 1)$  and  $\zeta_r(s) = \zeta_1(s - r + 1)$  if  $s \geq r - 1$ .

Clearly,  $\zeta_r$  have that same  $C^\infty$  norm for all  $r \geq 1$ .

Take coordinate system  $w = (x, y, t, z)$  as in (3.1). Then we define  $h : \Delta \rightarrow \Delta$  by

$$h(w) = T_{\theta(s, s', s'')}(w), \quad \theta(s, s', s'') = \theta \zeta_{s'/s}(x/s) \zeta_{s''/s}(y/s) \zeta_1(\sqrt{t^2 + z^2}/s).$$

By the construction, we see that  $h$  satisfies (a) and (b). For (c), note that if  $\theta = 0$ , then  $h = \text{id}$ , and note that the  $C^1$  norm of  $h$  change smoothly with  $\theta$ , we get the result.  $\square$

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#### REFERENCES

- [B] R. Bowen, *Symbolic Dynamics for Hyperbolic Flows*, Amer. J. Math., 95 1973, 429-460
- [BR] R. Bowen and D. Ruelle, *The Ergodic Theory of Axiom Flows*, Inventiones Math. **29** (1975), 181-202.
- [BPSW] K. Burns, C. Pugh, M. Shub and A. Wilkinson, *Recent results about stable ergodicity* in "Smooth Ergodic Theory and Its Applications (Seattle, 1999)", Proc. Sympos. Pure Math., **69**, Amer. Math. Soc., Providence, RI, 2001
- [CS] C.-Q. Cheng and Y.-S. Sun, *Existence of invariant tori in three dimensional measure-preserving mappings*, Celestial Mech. Dynam. Astronom. **47** (1989/90), 275-292.
- [DHP] D. Dolgopyat, H. Hu and Ya. Pesin, *An example of a smooth hyperbolic measure with countably many ergodic components*, in "Smooth Ergodic Theory and Its Applications (Seattle, 1999)", Proc. Sympos. Pure Math., **69**, Amer. Math. Soc., Providence, RI, 2001
- [NT] V. Nitică and A. Török, *An open dense set of stably ergodic diffeomorphisms in a neighborhood of a non-ergodic one*, Topology **40**(2001), 259-278.
- [P] Ya. Pesin, *characteristic Ljapunov exponents, and smooth ergodic theory*, Russian Math. Surveys **32** (1977), 55-114.



- [PS] C. Pugh and M. Shub, *Stably ergodic dynamical systems and partial hyperbolicity*, J. of Complexity **13** (1997), 125–179.
- [S] M. Shub, “Global Stability of Dynamical Systems”, Springer-Verlag, New York, 1987.
- [SW] M. Shub and A. Wilkinson, *Pathological foliations and removable zero exponents* Invent. Math. **139** (2000), 495–508.
- [X] Z. Xia, *Existence of invariant tori in volume-preserving diffeomorphisms*, Ergod. Th. and Dynam. Syst. **12** (1992), 621–631.

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