A VOLUME PRESERVING FLOW WITH ESSENTIAL COEXISTENCE OF ZERO AND NONZERO LYAPUNOV EXPONENTS

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Abstract. We demonstrate essential coexistence of hyperbolic and non-hyperbolic behavior in the continuous-time case by constructing a smooth volume preserving flow on a 5-dimensional compact smooth manifold that has nonzero Lyapunov exponents almost everywhere on an open and dense subset of positive but not full volume and is ergodic on this subset while having zero Lyapunov exponents on its complement. The latter is a union of 3-dimensional invariant submanifolds and on each of these submanifolds the flow is linear with Diophantine frequency vector.

1. Introduction

The goal of this paper is to extend the main result in [10] to dynamical systems with continuous time thus demonstrating coexistence of regular and chaotic dynamics in an "essential" way.

Theorem 1.1 (Main Theorem). *There exists a compact smooth Riemannian manifold $\mathcal{M}$ of dimension 5 and a C^∞ flow $h^t : \mathcal{M} \to \mathcal{M}$ such that*

(1) h^t preserves the Riemannian volume m on $\mathcal{M}$;
(2) h^t $(t \neq 0)$ has nonzero Lyapunov exponents (except for the exponent in the flow direction) almost everywhere on an open, dense and connected subset $\mathcal{U} \subset \mathcal{M}$; moreover, $h^t|\mathcal{U}$ is an ergodic flow;
(3) the complement $\mathcal{U}^c$ has positive volume and is a union of 3-dimensional invariant submanifolds; h^t is a non-identity linear flow with Diophantine frequency vector on each invariant submanifold and h^t has zero Lyapunov exponents on $\mathcal{U}^c$.

We stress that each of the 3-dimensional invariant submanifolds is in turn a union of 2-dimensional invariant tori on which h^t is a linear flow

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with Diophantine frequency vector (see Section 3 for details). This fact makes our construction nontrivial.[1]

We emphasize the requirement that the open set $\mathcal{U}$ is everywhere dense.[2] Donnay [8] constructed an example of a surface on which the geodesic flow exhibits the coexistence phenomenon. It is obtained by inserting a light-bulb cap into a negatively curved surface. In this example the set of geodesics, which are trapped in the cap, is invariant, has positive volume and almost every point in this set has zero Lyapunov exponents. Since it has non-empty interior, the stochastic sea in this example, i.e., the analog of the set $\mathcal{U}$ in our case, is not dense.

While this paper deals only with dynamical systems with continuous time, it is worth mentioned that in the discrete-time case essential coexistence of chaotic and regular behavior have been demonstrated in various situations by Przytycki and Liverani for area preserving diffeomorphisms and by Bunimovich for billiards; see the paper [5], which surveys recent result on essential coexistence, and references therein.

We split the proof of Theorem 1.1 into several steps. In Section 2 we present some background information and basic notations from the theory of partial hyperbolicity and in particular, introduce the notion of pointwise partial hyperbolicity on open sets for flows. In Section 3 we construct the manifold $\mathcal{M}$, the open set $\mathcal{U}$ and introduce the "start-up" flow f^t that satisfies the statements (3) of the theorem. In Section 4 we construct a volume preserving flow g^t, which is a small perturbation of f^t and does not affect the action of f^t on the set $\mathcal{U}^c$. The flow g^t has nonzero Lyapunov exponents on a subset of positive volume in $\mathcal{U}$. Finally, in Section 5 we construct the desired flow h^t as a small perturbation of the flow g^t. The proof of the Main Theorem is given in Section 6.

In our construction of the flows g^t and h^t we use the perturbation techniques developed in [10] for the case of diffeomorphisms (these techniques originated in [15], [7] and [6]). However, there is a crucial difference between the discrete-time and continuous-time cases. To

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[1] Indeed, consider the C^∞ volume preserving diffeomorphism P constructed in [10]. It has nonzero Lyapunov exponents almost everywhere on an open, dense and connected subset $\mathcal{U} \subset \mathcal{M}$ and $P|\mathcal{U}$ is ergodic. Furthermore, the complement $\mathcal{U}^c$ has positive volume, $P|\mathcal{U}^c$ is the identity map and has zero Lyapunov exponents. A special flow P^t over P has nonzero Lyapunov exponents (except for the exponent in the flow direction) almost everywhere on an open, dense and connected subset $\mathcal{U} \times [0,1]/\sim$ (where $\sim$ means that the points $(x,1)$ and $(Px,0)$ are identified) of not full volume, however, P^t is a periodic flow on its complement $\mathcal{U}^c \times [0,1]/\sim$.

[2] To some extent this justifies to view our example as of "KAM-type" despite the fact that the flow h^t is not close to a completely integrable one.

effect our construction we ought to make perturbations of the vector fields that generate the required flows but we want these perturbations to produce similar effects on the time-1 maps of the flows as in [10]. This is made possible due to the crucial fact that the "start-up" flow f^t has a global cross-section and in our construction we ensure that both perturbation flows g^t and h^t preserve this cross-section. We achieve this by using specific formulae for perturbations of the vector fields. At the core of our construction lies a new concept of pointwise partially hyperbolic flows on open sets. In Section 2 we introduce such flows and we study their ergodicity.

2. Preliminaries

See [2, 10, 13] for more details.

Consider a diffeomorphism f acting on a compact smooth Riemannian manifold $\mathcal{M}$. It is called *uniformly partially hyperbolic* on a compact invariant subset $\Lambda \subset \mathcal{M}$ if

(1) for every $x \in \Lambda$ the tangent space at x admits an invariant splitting

$$T_x\mathcal{M} = E^s(x) \oplus E^c(x) \oplus E^u(x) \tag{2.1}$$

into *stable* $E^s(x) = E^s_f(x)$, *central* $E^c(x) = E^c_f(x)$ and *unstable* $E^u(x) = E^u_f(x)$ subspaces;

(2) there are numbers $0 < \lambda < \widetilde{\lambda} \le 1 \le \widetilde{\mu} < \mu$ such that for every $t = 1, 2, \dots$

$$\begin{aligned} & \|df^t v\| \le \lambda^t \|v\|, && v \in E^s(x),\\ \widetilde{\lambda}^t\|v\| \le {} & \|df^t v\| \le \widetilde{\mu}^t\|v\|, && v \in E^c(x),\\ \mu^t\|v\| \le {} & \|df^t v\|, && v \in E^u(x). \end{aligned} \tag{2.2}$$

In this paper we need a weaker property than uniform partial hyperbolicity. Let $\mathcal{S} \subset \mathcal{M}$ be an invariant open subset. We say that a diffeomorphism F is *pointwise partially hyperbolic* on $\mathcal{S}$ if for every $x \in \mathcal{S}$ the tangent space at x admits an invariant splitting (2.1) and there are continuous functions $0 < \lambda(x) < \widetilde{\lambda}(x) \le 1 \le \widetilde{\mu}(x) < \mu(x)$ such that (2.2) holds with constants λ, $\widetilde{\lambda}$, $\widetilde{\mu}$ and μ replaced with these functions. Pointwise partially hyperbolic diffeomorphisms on compact manifolds were introduced in [4] where their ergodic properties were studied. If a diffeomorphism is pointwise partially hyperbolic on an open subset of a compact manifold then it could fail to have "nice" properties and in particular, could be not ergodic (see the discussion below).

We now consider a smooth flow f^t on $\mathcal{M}$ which is generated by the vector field $\mathcal{X}_f(x) = \frac{d}{dt}f^t(x)|_{t=0}$. We say that the flow is *uniformly partially hyperbolic* on a compact invariant subset $\Lambda \subset \mathcal{M}$ if for every $x \in \mathcal{M}$ the tangent space at x admits an invariant splitting (2.1) into stable $E^s(x) = E^s_f(x)$, central $E^c(x) = E^c_f(x)$ and unstable $E^u(x) = E^u_f(x)$ subspaces such that the vector field $\mathcal{X}_f(x)$ is contained in the central subspace $E^c(x)$ and there are numbers $0 < \lambda < \widetilde{\lambda} \le 1 \le \widetilde{\mu} < \mu$ such that (2.2) holds for all $t \in [0,1]$. Note that if a flow f^t is uniformly partially hyperbolic, then for every $t \neq 0$ the time-t map is uniformly partially hyperbolic with the same invariant splitting.

Given an invariant open subset $\mathcal{S} \subset \mathcal{M}$ we call a flow f^t *pointwise partially hyperbolic* on $\mathcal{S}$ if its time-1 map f^1 is pointwise partially hyperbolic on $\mathcal{S}$.

Given $\delta > 0$, we say that a flow g^t is (C^1, δ)-close to f^t on an invariant set Λ if $\mathcal{X}_g = \mathcal{X}_f$ outside Λ and $\|\mathcal{X}_g - \mathcal{X}_f\| \le \delta$. Uniformly partially hyperbolic flows form an open set in the C^1 topology (see Lemma B.8 in the Appendix B).

Given a an open subset $\mathcal{S} \subset \mathcal{M}$, we call a partition $\mathcal{P}$ of $\mathcal{S}$ a (δ, q)-*foliation with smooth leaves* if there exist continuous functions $\delta = \delta(x) > 0$, $q = q(x) > 0$, and an integer $k > 0$ such that for each $x \in \mathcal{S}$:

(1) There exists a smooth immersed k-dimensional manifold $W(x)$ containing x for which $\mathcal{P}(x) = W(x)$ where $\mathcal{P}(x)$ is the element of the partition $\mathcal{P}$ containing x. The manifold $W(x)$ is called the *global leaf* of the foliation at x; the connected component of the intersection $W(x) \cap B(x, \delta(x))$ that contains x is called the *local leaf* at x and is denoted by $V(x)$;
(2) There exists a continuous map $\phi_x : B(x, q(x)) \to C^1(D, \mathcal{M})$ (where D is the unit ball) such that $V(y)$ is the image of the map $\phi_x(y) : D \to \mathcal{M}$ for each $y \in B(x, q(x))$; the number $q(x)$ is called the *size* of $V(x)$.

We say that a foliation with smooth leaves is *absolutely continuous* if for almost every $x \in \mathcal{S}$ and almost every $y \in B(x, q(x))$ the conditional measure on the local leaf $V(y)$, generated by the volume m on $\mathcal{M}$ and the partition of $B(x, q(x))$ by the local leaves, is absolutely continuous with respect to the leaf volume $m_{V(y)}$ on $V(y)$. [3]

Let W_1 and W_2 be two foliations of $\mathcal{S}$ with smooth leaves that are transversal to each other at every point $z \in \mathcal{S}$. Let also $\mathcal{S}_1 \subset \mathcal{S}$ be an open subset. We say that the pair W_1 and W_2 has the *accessibility*

[3] The leaf volume $m_{V(y)}$ is generated by the restriction of the Riemannian metric on $\mathcal{M}$ to the smooth submanifod $V(y)$.

property on $\mathcal{S}_1$ if any two points $z, z' \in \mathcal{S}_1$ are *accessible* via a (u,s)-path in $\mathcal{S}$, that is

(1) there exists a collection of points $z_1, \dots, z_n \in \mathcal{S}$ such that $z = z_1$, $z' = z_n$ and $z_k \in W_i(z_{k-1})$ for $i = 1, 2$ and $k = 2, \dots, n$;
(2) the points z_{k-1} and z_k can be connected by a smooth curve $\gamma_k \subset V_i(z_{k-1})$ in $\mathcal{S}$ for $i = 1$ or 2 and $k = 2, \dots, n$. [4]

The collection of the leaf-wise paths γ_k is called a (u,s)-*path* and is denoted by $[z_1, \dots, z_n]$.

For a uniformly partially hyperbolic flow f^t one can construct stable and unstable local manifolds of uniform size at every point in Λ. This may not be true for a flow that is pointwise partially hyperbolic on an open set $\mathcal{S}$. However, all pointwise partially hyperbolic flows that we consider in this paper will have global stable and unstable transverse foliations with smooth leaves. We denote these foliations by $W^s = W^s_f$ and $W^u = W^u_f$ respectively.

More precisely, let f^t be a flow that is pointwise partially hyperbolic on an open set $\mathcal{S}$ and let g^t be a sufficiently small perturbation of f^t in the C^1 topology.

Definition 2.1. *We call the perturbation g^t* gentle *if there exists an open set $\mathcal{U} \subset \mathcal{S}$ such that $\bar{\mathcal{U}} \subset \mathcal{S}$, $\mathcal{U}$ is invariant under both f^t and g^t and $f^t|\mathcal{U}^c = g^t|\mathcal{U}^c$.*

Theorem 2.2. *Assume that the strongly stable and unstable subspaces $E^s_{f^t}$ and $E^u_{f^t}$ for f^t are integrable to continuous strongly stable and unstable foliations $W^s_{f^t}$ and $W^u_{f^t}$ respectively with smooth leaves and that these foliations are transverse. Then for any sufficiently small gentle perturbation g^t in the C^1 topology the strongly stable and unstable subspaces $E^s_{g^t}$ and $E^u_{g^t}$ for g^t are integrable to continuous strongly stable and unstable foliations $W^s_{g^t}$ and $W^u_{g^t}$ respectively with smooth leaves and these foliations are transverse.*

We call a flow f^t that is pointwise partially hyperbolic on an open set $\mathcal{S}$ *dynamically coherent* if the subbundles $E^{cu} = E^c \oplus E^u$, E^c, and $E^{cs} = E^c \oplus E^s$ are integrable to continuous foliations with smooth leaves W^{cu}, W^c and W^{cs}, called respectively the *center-unstable*, *center* and *center-stable foliations*. Furthermore, the foliations W^c and W^u are subfoliations of W^{cu}, while W^c and W^s are subfoliations of W^{cs}.

The following result is an extension of the classical result in [9, 14] to the case of flows that are pointwise partially hyperbolic on an open

[4]We stress that $V_i(z_{k-1})$ is the *local* leaf of W_i at z_i. In particular, the length of the curve γ_k (the *leg* of the path) does not exceed $\delta(z_{k-1})$.

subset. It shows that dynamical coherence is a robust property within the class of gentle perturbations. The proof of this result is a simple modification of the argument in [9].

Theorem 2.3. *Suppose that f^t is a flow that is pointwise partially hyperbolic on an open set $\mathcal{S}$. Assume that f^t possesses transverse strongly stable and unstable foliations with smooth leaves. Assume also that the center distribution is integrable to a smooth center foliation W^c. Then f^t is dynamically coherent. Moreover, any flow that is close to f^t in the C^1 topology and is a gentle perturbation of f^t is dynamically coherent.*

Since both subbundles E^{cu} and E^{cs} vary continuously with the map, so does E^c and the corresponding center foliation W^c.

Given a smooth flow f^t, we denote by

$$\lambda(x,v) = \lambda(x,v,f^t) = \limsup_{t\to\infty} \frac{1}{t} \log \|df^t v\|$$

the *Lyapunov exponent* of a nonzero vector v at $x \in \mathcal{M}$ and by $\lambda_i(x) = \lambda_i(x,f^t)$, $i = 1, \dots, \dim \mathcal{M}$, the values of the Lyapunov exponents at x in the decreasing order. We also denote by

$$L_k(f^t) = \int_{\mathcal{M}} \sum_{i=1}^{k} \lambda_i(x,f^t) dm(z), \tag{2.3}$$

where m is the Riemannian volume. We call this number the k-th average Lyapunov exponent of f^t.

Consider a C^2 flow f^t of a compact smooth manifold $\mathcal{M}$ that is pointwise partially hyperbolic on an open invariant set $\mathcal{S}$. Assume that f^t preserves a smooth measure on $\mathcal{M}$. We say that f^t has *positive central exponents* if there is an invariant set $\mathcal{A} \subset \mathcal{S}$ of positive measure such that for every $x \in \mathcal{A}$ and every $v \in E^c(x) \setminus \mathrm{Span}\{\mathcal{X}_f(x)\}$ the Lyapunov exponent $\lambda(x,v) > 0$. The following theorem plays an important role in the proof of our Main Theorem.

Theorem 2.4. *Assume that the following conditions hold:*

(1) f^t has strongly stable and unstable (δ,q)-foliations W^s and W^u where $\delta = \delta(x)$ and $q = q(x)$ are continuous functions on $\mathcal{S}$;
(2) the foliations W^s and W^u are absolutely continuous;
(3) f^t has the accessibility property via the foliations W^s and W^u on $\mathcal{S}$;
(4) f^t has positive central exponents;
(5) The Lyapunov exponents in the stable subspace $E^s(x)$ are all negative and the Lyapunov exponents in the unstable subspace $E^u(x)$ are all positive for almost every x.

Then f^t has positive central exponents at almost every point $x \in \mathcal{S}$ and $f^t|\mathcal{S}$ is an ergodic flow.

Proof. Following Theorem 2.2 in [10], we first show that f^t has positive central exponents at almost every point $x \in \mathcal{S}$ and $f^t|\mathcal{S}$ is an ergodic flow.

There is a set $\mathcal{A} \subset \mathcal{S}$ of positive measure such that the flow $f^t|\mathcal{A}$ has nonzero Lyapunov exponents except along the flow direction. Hence, it has at most countably many ergodic components of positive measure in $\mathcal{A}$ (see [2]). Each such component contains the set

$$A(x) = \bigcup_{y \in V^{cu}(x)} V^s(y),$$

where x is the density point of $\mathcal{A}$ and $V^{cu}(x)$ is a center-unstable local manifold at x. Since the strong stable foliation W^s is continuous, the set $A(x)$ is open (mod 0) in $\mathcal{M}$[5] and hence, the set $\mathcal{A}$ is itself open (mod 0). We will show that the trajectory of almost every point in $\mathcal{S}$ is dense, which yields that $\mathcal{A} = \mathcal{S}$ (mod 0) and that $f^t|S$ is ergodic.

The proof of this claim for partially hyperbolic diffeomorphisms is given in [1] and can be extended to our case literally. We present the argument here for the reader's convenience. We call a point p *good* for a given open set U if p has a neighborhood in which the orbit of almost every point enters U. It suffices to show that an arbitrary point p is good. Since f^t is accessible, there is a (u, s)-path $[z_0, \ldots, z_k]$ with $z_0 \in U$ and $z_k = p$. We will show by induction on j that each point z_j is good. This is obvious for $j = 0$. Now suppose that z_j is good, then z_j has a neighborhood N such that $\mathrm{Orb}(x) \cap U \neq \emptyset$ for almost every $x \in N$. Let B be the subset of N consisting of points with this property that are also both forward and backward recurrent. It follows from the Poincaré recurrence theorem that B has full measure in N. If $x \in B$, any point $y \in W^s(x) \cup W^u(x)$ has the property that $\mathrm{Orb}(y) \cap U \neq \emptyset$. The absolute continuity of the foliations W^s and W^u means that the set

$$\bigcup_{x \in B} W^s(x) \cup W^u(x)$$

has full measure in the set

$$\bigcup_{x \in N} W^s(x) \cup W^u(x).$$

The latter is a neighborhood of z_{j+1}. Hence, z_{j+1} is good. □

[5]That is there is an open set $\mathcal{V} \subset \mathcal{M}$ such that $A(x) = \mathcal{V}$ (mod 0) with respect to the volume m.

3. Construction of the "start-up" flow f^t

Let A be an Anosov automorphism of the 2-torus $X = \mathbb{T}^2$ with expanding rate $\eta > 1$ along the unstable direction. Consider the suspension flow S^t on the suspension manifold $\mathcal{N} = X \times \mathbb{R}/\sim$, with the identification $(x, \tau + 1) \sim (Ax, \tau)$. The action of the suspension flow on $\mathcal{N}$ is exactly $S^t(x, \tau) = (x, \tau + t)$. See Appendix A for more details of the geometric structure of $\mathcal{N}$.

Given $\alpha \in \mathbb{T}^2$, let $T^t_\alpha : \mathcal{N} \to \mathcal{N}$ be a linear flow defined by $(x, \tau) \mapsto (x + t\alpha, \tau)$. It preserves each level set $X \times \{\tau\}$.

Set $Y = \mathbb{T}^2$ and $\mathcal{M} = \mathcal{N} \times Y$. To effect our construction we choose:

(A1) a Cantor set $C \subset Y$ of positive area whose complement $U = Y \backslash C$ is a non-empty open and connected set;
(A2) an open square U_0 such that $\overline{U_0} \subset U$;
(A3) an open neighborhood U_1 of $\overline{U_0}$ such that $\overline{U_1} \subset U$, whose choice will be specified in Section 5.2.1.

We also choose a C^∞ function $\kappa : Y \to \mathbb{R}$ such that

(κ1) $\kappa(y) > 0$ for $y \in U$ and $\kappa(y) = 0$ for $y \in C$;
(κ2) $\kappa(y) = 1$ for $y \in U_1$;
(κ3) $\|\kappa\|_{C^1} \le 1$

(the existence of such a function κ follows from the specific construction of the Cantor set and the choice of the set U_1 in Section 5.1) and a C^∞ map $\alpha : Y \to \mathbb{R}^2$ such that

(α1) $\alpha(y) = 0$ for $y \in U_1$;
(α2) $\alpha(y) = \alpha_0 > 0$ for all $y \in C$ where α_0 is a Diophantine vector;
(α3) $\sup_{y \in Y} \|\alpha(y)\| \le \bar{\alpha}$, where $\bar{\alpha}$ is a positive number determined in Section 5.2.

We now set $\mathcal{U} = \mathcal{N} \times U$ and $\mathcal{U}^c = \mathcal{N} \times C$ and define the flow f^t on $\mathcal{M}$ by the formula

$$f^t((x, \tau), y) = ((x + t\alpha(y), \tau + t\kappa(y)), y) \tag{3.1}$$

where $(x, \tau) \in \mathcal{N}$ and $y \in Y$. The following proposition describes the properties of the flow f^t and its proof follows immediately from the definitions.

Proposition 3.1. *The following statements hold:*

(1) f^t is a C^∞ volume preserving flow;
(2) f^t preserves each fiber $\mathcal{N} \times \{y\}$, on which f^t is the composition of the scaled suspension flow $S^{t\kappa(y)}$ and the linear flow $T^t_{\alpha(y)}$; in

particular, f^t is exactly the suspension flow S^t on $\mathcal{N} \times \{y\}$ for $y \in \overline{U_1}$;

(3) *f^t is pointwise partially hyperbolic on $\mathcal{U}$ with one-dimensional stable E^s_f, one-dimensional unstable E^u_f and 3-dimensional center E^c_f subbundles; E^s_f and E^u_f are integrable to strongly stable and unstable foliations W^s_f and W^u_f with smooth leaves, which are absolutely continuous, uniformly transversal and have local leaves of uniform size;*

(4) *f^t is uniformly partially hyperbolic on $\mathcal{N} \times A$ where $A \subset U$ is a subset, and hence f^t is dynamically coherent with the central foliation $W^c_f = W^c_{S^t} \times Y$;*

(5) *f^t preserves every 2-dimensional torus $X \times \{\tau\} \times \{y\}$ ($\tau \in [0,1]$, $y \in C$ are fixed) and acts on it as a linear flow with a Diophantine frequency vector; moreover, $f^t|\mathcal{U}^c$ has all zero Lyapunov exponents on $\mathcal{U}^c$;*

(6) *for every $z = ((x,\tau),y) \in \mathcal{M}$ the Lyapunov exponents of f^t are as follows:*

$$\lambda_1(z,f^t) = \kappa(y)\log\eta \geq 0 = \lambda_2(z,f^t) = \lambda_3(z,f^t) = \lambda_4(z,f^t)$$
$$\geq \lambda_5(z,f^t) = -\kappa(y)\log\eta;$$

moreover, if $z \in \mathcal{U}$, then $\lambda_1(z,f^t) = \lambda^u(z,f^t) > 0$ is the Lyapunov exponent in the $E^u_f(z)$ subspace, $\lambda_5(z,f^t) = \lambda^s(z,f^t) < 0$ is the Lyapunov exponent in the $E^s_f(z)$ subspace, and $\lambda_2(z,f^t)$, $\lambda_3(z,f^t)$ and $\lambda_4(z,f^t)$ are Lyapunov exponents in the flow direction and two directions in Y respectively.

4. Removing zero exponents

In this section we will construct a gentle perturbation g^t of the original flow f^t with positive central Lyapunov exponents on a set of positive volume but not necessarily ergodic. Then we will perturb g^t to the desired flow h^t of the main theorem in Section 5.

Given $z \in \mathcal{M}$, there is a local Cartesian coordinate system (u,s,τ,a,b) (see Appendix A) such that

$$F^u(z) := \frac{\partial}{\partial u} = E^u_f(z), \quad F^s(z) := \frac{\partial}{\partial s} = E^s_f(z), \quad F^\tau(z) := \frac{\partial}{\partial \tau} = E^\tau_f(z)$$

are the unstable, stable and flow directions of f^t respectively, and

$$F^a(z) := \frac{\partial}{\partial a} = E^a_f(z), \qquad F^b(z) := \frac{\partial}{\partial b} = E^b_f(z)$$

are the other two central directions tangent to Y.

The following statement describes properties of the flow g^t.

Proposition 4.1. *Given $\delta_g > 0$, there is a C^∞ volume preserving flow g^t on $\mathcal{M}$ such that the following hold.*

(1) g^t is (C^1, δ_g)-close to f^t, i.e., $\|\mathcal{X}_f - \mathcal{X}_g\| \le \delta_g$, where $\mathcal{X}_f$ and $\mathcal{X}_g$ are the vector fields of the flows f^t and g^t respectively.

(2) $g^t = f^t$ outside $\mathcal{N} \times U_0$, and hence g^t is a gentle perturbation of f^t and satisfies Statements (3)-(5) of Proposition 3.1.

(3) g^t preserves the subbundles E_f^ω, $\omega = uab, uab\tau$; moreover,

$$\det(dg^t|E_f^\omega(z)) = \det(df^t|E_f^\omega(z)), \text{ for all } z \in \mathcal{M}. \tag{4.1}$$

(4) The average Lyapunov exponents of g^t satisfy

$$L_5(g^t) = 0 < L_1(g^t) < L_2(g^t) < L_3(g^t) = L_4(g^t). \tag{4.2}$$

To prove this proposition, we extend the approach in [10] to the case of flows and obtain the flow g^t as a result of two consecutive perturbations. First, we perturb the start-up flow f^t to a flow $\widetilde{g}^t$ by adding a *rotational vector field* $\widetilde{\mathcal{X}}_R$ to the vector field $\mathcal{X}_f$.[6] This produces two positive average Lyapunov exponents for the flow $\widetilde{g}^t$ in the E_f^{ua} subbundle, i.e., $L_1(\widetilde{g}^t) < L_2(\widetilde{g}^t)$. Next, we perturb $\widetilde{g}^t$ to the desired flow g^t by adding another rotational vector field $\mathcal{X}_R$ to the vector field $\mathcal{X}_{\widetilde{g}}$ for the flow $\widetilde{g}^t$. As a result the flow g^t has three positive average Lyapunov exponents in the E_f^{uab} subbundle, i.e., $L_1(g^t) < L_2(g^t) < L_3(g^t)$.

The vector fields $\widetilde{\mathcal{X}}_R$ and $\mathcal{X}_R$ are chosen to be supported on disjoint open subsets $\widetilde{\Omega}_R$ and Ω_R of $\mathcal{N} \times U_0$ respectively such that $\widetilde{\mathcal{X}}_R = 0$ outside $\widetilde{\Omega}_R$, $\mathcal{X}_R = 0$ outside Ω_R and $\|\widetilde{\mathcal{X}}_R\|_{C^1}$, $\|\mathcal{X}_R\|_{C^1} < \delta_g/2$. Since $\mathcal{N} \times U_0$ is invariant under f^t, we have that $g^t = \widetilde{g}^t = f^t$ outside $\mathcal{N} \times U_0$.

Our construction utilizes the following crucial feature of the flow f^t: the set

$$\Pi_0 = X \times \{0\} \times U_0 \tag{4.3}$$

is a global cross-section of $f^t|\mathcal{N} \times U_0$, and the time-1 map restricted to Π_0 is exactly the Poincaré return map of f^t to Π_0. Furthermore, we make the construction of vector fields $\widetilde{\mathcal{X}}_R$ and $\mathcal{X}_R$ in such a way that Π_0 is also a global cross-section for both flows $\widetilde{g}^t$ and g^t with the time-1 maps to be the Poincaré return map to Π_0. This fact allows us to apply arguments similar to those in [10] to our flow case by focusing on the time-1 maps.

[6] That is the flow generated by $\widetilde{\mathcal{X}}_R$ is concentrated in a small neighborhood of a point in $\mathcal{M}$ where it acts as a small rotation around this point; see (4.6).

4.1. **Construction of the flow $\widetilde{g}^t$.** In this section we construct the flow $\widetilde{g}^t$ by perturbing the vector field $\mathcal{X}_f$ inside the set $\mathcal{N} \times U_0$.

To effect our construction we choose distinct periodic points q, p^a, p^b and p^τ of the Anosov automorphism A of X, which are close to each other. Let $V^s_A(q)$, $V^u_A(q)$, $V^s_A(p^i)$ and $V^u_A(p^i)$, $i = a, b, \tau$ be the stable and unstable local manifolds at these periodic points. We may assume that each intersection $V^u_A(q) \cap V^s_A(p^i)$ and $V^u_A(p^i) \cap V^s_A(q)$ consists of a single point, which we denote by $[q, p^i]$ and $[p^i, q]$ respectively. Let γ^i denote the closed quadrilateral path with the collection of points q, $[q, p^i]$, p^i, $[p^i, q]$ and q, and let

$$\gamma(q) = V^u_A(q) \cup V^s_A(q), \quad \gamma(p^i) = V^u_A(p^i) \cup V^s_A(p^i).$$

Choose $\nu > 0$ and set for $i = a, b, \tau$,

$$\begin{aligned} \Omega^i_0(\nu) =& \Big(\bigcup_{t \in [0, \iota(p^i)]} B_{\mathcal{N}}(f^t(p^i, 0), \nu)\Big) \times U_0, \\ \widehat{\Omega}^i_0(\nu) =& \Big(\bigcup_{(x,\tau) \in (\gamma(q) \times [0, \iota(q)]) \cup (\gamma(p^i) \times [0, \iota(p^i)])} B_{\mathcal{N}}((x, \tau), \nu)\Big) \times U_0, \\ \Omega_0 =& \Omega_0(\nu) = \Big(\bigcup_{i = a, b, \tau} \Omega^i(\nu)\Big) \cup \Big(\bigcup_{i = a, b, \tau} \widehat{\Omega}^i(\nu)\Big), \end{aligned} \tag{4.4}$$

where $\iota(q)$ and $\iota(p^i)$ are the periods of q and p^i respectively, and $B_{\mathcal{N}}((x, \tau), r)$ is the ball in $\mathcal{N}$ of radius r centered at the point $(x, \tau) \in \mathcal{N}$. We choose a sufficiently small number ν to ensure that

$$m(\mathrm{Proj}_{\Pi_0} \Omega_0) \le 0.05 m(\Pi_0),$$

where Π_0 is given by (4.3), and $\mathrm{Proj}_{\Pi_0} : \mathcal{N} \times U_0 \to \Pi_0$ is the natural projection onto Π_0 given by the formula

$$\mathrm{Proj}_{\Pi_0}((x, \tau), y) = ((x, 0), y).$$

To construct the vector field $\widetilde{\mathcal{X}}_R$ we choose a C^∞ function $\psi : \mathbb{R} \to [0, 1]$ such that

1. (1) $\psi = 1$ on $(-0.9, 0.9)$;
2. (2) $\psi > 0$ on $(-1, 1)$ and $\psi = 0$ outside $(-1, 1)$;
3. (3) $\|\psi\|_{C^1} \le 10$.

Observe that $\mathcal{N} \times U_0$ is invariant under the flow f^t and that

$$f^t((x, \tau), y) = ((x, \tau + t), y)$$

for $((x, \tau), y) \in \mathcal{N} \times U_0$. In other words, $f^t | \mathcal{N} \times U_0$ is the product of the suspension flow on $\mathcal{N}$ and the identity map on U_0. It follows that Π_0 is a global cross-section for the flow $f^t | \mathcal{N} \times U_0$, and the time-1 map restricted to Π_0 is $f^1 = A \times Id$ and is exactly the Poincaré return map of f^t to Π_0. Given a subset $\Pi \subset \Pi_0$, we call a set $\Pi \times [\tau_1, \tau_2] \subset \Pi_0 \times \mathbb{R}/\sim = \mathcal{N} \times U_0$ a *tube* if

$$f^t(\Pi \times \{\tau_1\}) \cap (\Pi \times \{\tau_1\}) = \emptyset \quad \text{for all } t \in [0, \tau_2 - \tau_1].$$

It is easy to check that $\Pi \times [\tau_1, \tau_2]$ is a tube if and only if the sets $\Pi, f^1(\Pi), f^2(\Pi), \dots, f^l(\Pi)$ are pairwise disjoint, where $l = \lfloor \tau_2 - \tau_1 \rfloor$. Choose a non-periodic point $z_0 = (x_0, 0, y_0) \in \Pi_0 \backslash \mathrm{Proj}_{\Pi_0} \Omega_0$, where x_0 is a non-periodic point of the Anosov automorphism A, y_0 is the center of the square U_0 and Ω_0 is the set given by (4.4).

In what follows we will use the local Cartesian coordinate system in a neighborhood of Π_0 originated at z_0 given by (u, s, t, a, b) where (u, s) are the coordinates in X along the stable and unstable directions of the hyperbolic diffeomorphism A, t is the coordinate along the time direction and (a, b) are coordinates in Y. In this coordinate system a point $z \in \Pi_0$ is given as $z = (u, s, 0, a, b)$. We will also use the ua-cylindrical coordinates (r, θ, τ, s, b), where $u = r\cos\theta$, $a = r\sin\theta$. Given $\varepsilon > 0$, one can choose a ua-cylinder $B \subset \Pi_0$ centered at z_0 of size ε, i.e.,

$$B = \{(r, \theta, 0, s, b) : r \le \varepsilon, |s| \le \varepsilon, |b| \le \varepsilon\}.$$

Given a sufficiently large $N_0 \ge 20k_0$ (the number k_0 is defined below in Lemma 4.5), we can choose ε so small that $f^i(B) \cap B = \emptyset$ for $i = 1, \dots, N_0$. Consider the tube

$$\widetilde{\Omega}_R = B \times [0, 1/2]. \tag{4.5}$$

Since $z_0 \notin \mathrm{Proj}_{\Pi_0}\Omega_0$, we can further reduce ε to ensure that $B \cap \mathrm{Proj}_{\Pi_0}(\Omega_0) = \emptyset$. Hence, $\widetilde{\Omega}_R \cap \Omega_0 = \emptyset$.

Given $\beta > 0$, define a C^∞ rotational vector field $\widetilde{\mathcal{X}}_R = \widetilde{\mathcal{X}}_{R,\beta}$ on $\mathcal{M}$ as follows:

$$\widetilde{\mathcal{X}}_{R,\beta}(z) = \begin{cases} \beta\widetilde{\psi}(z)\dfrac{\partial}{\partial\theta}, & z \in \widetilde{\Omega}_R, \\ \\ 0, & z \in \mathcal{M}\backslash\widetilde{\Omega}_R, \end{cases} \tag{4.6}$$

where

$$\widetilde{\psi}(z) = \widetilde{\psi}(r, \theta, \tau, s, b) = \psi\left(\frac{r^2}{\varepsilon^2}\right)\psi\left(\frac{s}{\varepsilon}\right)\psi\left(\frac{b}{\varepsilon}\right)\psi\left(\frac{\tau - (1/4)}{1/4}\right).$$

It is easy to see that $\|\widetilde{\psi}\dfrac{\partial}{\partial\theta}\| \le c$ where $c > 0$ is a constant, which is independent of ε. Hence, $\|\widetilde{\mathcal{X}}_{R,\beta}\| \to 0$ as $\beta \to 0$. Furthermore, $\widetilde{\mathcal{X}}_{R,\beta}$ is divergence free. Let $\widetilde{g}^t = \widetilde{g}^t_\beta$ be the flow generated by the vector field

$$\mathcal{X}_{\widetilde{g}} = \mathcal{X}_{\widetilde{g},\beta} = \mathcal{X}_f + \widetilde{\mathcal{X}}_{R,\beta}.$$

Proposition 4.2. *There exists $\beta > 0$ such that $\widetilde{g}^t = \widetilde{g}^t_\beta$ is a C^∞ volume preserving flow with the following properties:*

(1) $\widetilde{g}^t$ is $(C^1, \delta_g/2)$-close to f^t, i.e., $\|\mathcal{X}_f - \mathcal{X}_{\widetilde{g}}\|_{C^1} \le \delta_g/2$, where $\mathcal{X}_f$ and $\mathcal{X}_{\widetilde{g}}$ are the vector fields corresponding to flows f^t and $\widetilde{g}^t$ respectively;

(2) $\widetilde{g}^t = f^t$ outside $\mathcal{N} \times U_0$, and hence $\widetilde{g}^t$ is a gentle perturbation of f^t and satisfies Statements (3)-(5) of Proposition 3.1;

(3) $\widetilde{g}^t$ preserves the subbundles E^ω_f, $\omega = ua, uab, uab\tau$; moreover,

$$\det(d\widetilde{g}^t | E^\omega_f(z)) = \det(df^t | E^\omega_f(z)) \quad \text{for all } z \in \mathcal{M}; \tag{4.7}$$

(4) the average Lyapunov exponents of $\widetilde{g}^t$ satisfy

$$L_5(\widetilde{g}^t) = 0 < L_1(\widetilde{g}^t) < L_2(\widetilde{g}^t) = L_3(\widetilde{g}^t) = L_4(\widetilde{g}^t); \tag{4.8}$$

(5) Π_0 is a global cross-section of the flow $\widetilde{g}^t | \mathcal{N} \times U_0$, and the time-1 map $\widetilde{g}^1$ is the Poincaré return map of g^t to Π_0; furthermore, there exist $\lambda > 0$ and a $\widetilde{g}^1$-invariant subset $\Pi \subset \Pi_0$ such that

$$m(\Pi) \ge 20k_0 m(\Pi \cap B) > 0$$

and for any $z \in \Pi$ the flow $\widetilde{g}^t$ has two positive Lyapunov exponents $\lambda_1(z, \widetilde{g}^t) > \lambda_2(z, \widetilde{g}^t) \ge \lambda$ along the E^{ua}_f subbundle.

Proof. Statements (1) and (2) are easy corollaries of the construction of the flow $\widetilde{g}^t$. To prove Statement (3) we will first show that $d\widetilde{g}^t$ preserves the subbundles E^{ua}_f. It suffices to check that for any smooth vector field $\mathcal{X} \in E^{ua}_f$ and any $z \in \widetilde{\Omega}_R$, the Lie bracket $[\mathcal{X}_{\widetilde{g}}(z), \mathcal{X}(z)] \in E^{ua}_f(z)$. Indeed, we have

$$\mathcal{X}_{\widetilde{g}}(z) = \frac{\partial}{\partial \tau} + \beta\widetilde{\psi}(z)\frac{\partial}{\partial \theta} = \frac{\partial}{\partial \tau} + \beta\widetilde{\psi}(z)\left(-a\frac{\partial}{\partial u} + u\frac{\partial}{\partial a}\right),$$

and the direct calculation yields

$$\left[\mathcal{X}_{\widetilde{g}}, \frac{\partial}{\partial \omega}\right] = \beta\left(\frac{\partial(a\widetilde{\psi})}{\partial \omega}\frac{\partial}{\partial u} - \frac{\partial(u\widetilde{\psi})}{\partial \omega}\frac{\partial}{\partial a}\right) \in E^{ua}_f, \quad \omega = u, a.$$

Since $d\widetilde{g}^t$ preserves the subbundle E^{ua}_f, it also preserves the subbundles E^{uab}_f and $E^{uab\tau}_f$. Next, consider the variational differential equations

$$\frac{d}{dt}f^t = D\mathcal{X}_f df^t, \qquad \frac{d}{dt}\widetilde{g}^t = D\mathcal{X}_{\widetilde{g}} d\widetilde{g}^t.$$

The determinants along E_f^ω with $\omega = ua, uab, uab\tau$ satisfy

$$\begin{aligned} \frac{d}{dt}\det(df^t|E_f^\omega) &= \operatorname{div}(\mathcal{X}_f|E_f^\omega)\det(df^t|E_f^\omega), \\ \frac{d}{dt}\det(d\widetilde{g}^t|E_f^\omega) &= \operatorname{div}(\mathcal{X}_{\widetilde{g}}|E_f^\omega)\det(d\widetilde{g}^t|E_f^\omega). \end{aligned} \tag{4.9}$$

Direct calculations show that $\widetilde{\mathcal{X}}_R = \mathcal{X}_{\widetilde{g}} - \mathcal{X}_f$ is divergence free along E_f^ω and thus $\operatorname{div}(\mathcal{X}_f|E_f^\omega) = \operatorname{div}(\mathcal{X}_{\widetilde{g}}|E_f^\omega)$. Therefore, using (4.9) and the fact that $\det(dg^0|E_f^\omega) = \det(df^0|E_f^\omega) = 1$ we find that

$$\det(d\widetilde{g}^t|E_f^\omega(z)) = \det(df^t|E_f^\omega(z))$$

and Statement (3) follows.

It remains to prove Statements (4) and (5). We need the following lemma showing that Π_0 is a global cross-section for the flow $\widetilde{g}^t|\mathcal{N} \times U_0$.

Lemma 4.3. *Given $z \in B$, the r-, s-, b- and τ- coordinates of $\widetilde{g}^t(z)$ and $f^t(z)$ are the same for $t \in [0, 1/2]$. Consequently, $\widetilde{g}^{\frac{1}{2}}(B) = f^{\frac{1}{2}}(B)$ and Π_0 is a global cross-section for the flow $\widetilde{g}^t|\mathcal{N} \times U_0$.*

Proof of the lemma. Let us compare the orbit segments of $\widetilde{g}^t(z)$ and $f^t(z)$ for $t \in [0, 1/2]$. Note that for any smooth function φ and any vector field $\mathcal{X}$ we have that

$$\frac{d}{dt}\varphi(F^t(z)) = L_{\mathcal{X}}\varphi|_{F^t(z)},$$

where $L_{\mathcal{X}}(\cdot)$ is the Lie derivative and F^t is the flow that is generated by $\mathcal{X}$. This implies that for $\omega = r, s, b$,

$$\begin{aligned} \frac{d}{dt}\tau(\widetilde{g}^t(z)) &= L_{\mathcal{X}_{\widetilde{g}}}\tau = \frac{\partial\tau}{\partial\tau} + \beta\widetilde{\psi}(\widetilde{g}^t(z))\frac{\partial\tau}{\partial\theta} = 1 = L_{\mathcal{X}_f}\tau = \frac{d}{dt}\tau(f^t(z)), \\ \frac{d}{dt}\omega(\widetilde{g}^t(z)) &= L_{\mathcal{X}_{\widetilde{g}}}\omega = \frac{\partial\omega}{\partial\tau} + \beta\widetilde{\psi}(\widetilde{g}^t(z))\frac{\partial\omega}{\partial\theta} = 0 = L_{\mathcal{X}_f}\omega = \frac{d}{dt}\omega(f^t(z)). \end{aligned}$$

Under the same initial condition at $t = 0$, we get that the r-, s-, b- and τ-coordinates of $\widetilde{g}^t(z)$ and $f^t(z)$ are the same. Since B has the cylindrical structure, we obtain that $\widetilde{g}^t(z) \in B \times \tau(f^t(z)) = B \times \{t\}$. In particular, $\widetilde{g}^{\frac{1}{2}}(B) = f^{\frac{1}{2}}(B) = B \times \{\frac{1}{2}\}$. Since $\mathcal{X}_{\widetilde{g}} = \mathcal{X}_f$ outside $\widetilde{\Omega}_R = B \times [0, 1/2]$, we have that $\widetilde{g}^1(\Pi_0) = f^1(\Pi_0) = \Pi_0$. In other words, Π_0 is also a global cross-section for $\widetilde{g}^t|\mathcal{N} \times U_0$. □

It follows from the lemma that Π_0 is a global cross-section for the flow $\widetilde{g}^t|\mathcal{N} \times U_0$ and the time-1 map $\widetilde{g}^1$ restricted to Π_0 is the Poincaré return map of $\widetilde{g}^t$ on Π_0. Therefore, (4.8) is equivalent to

$$L_4(\widetilde{G}) = 0 < L_1(\widetilde{G}) < L_2(\widetilde{G}) = L_3(\widetilde{G}), \tag{4.10}$$

where $\widetilde{G} = \widetilde{G}_\beta = \widetilde{g}^1|\Pi_0$. In fact, by (4.7), we have that $L_k(\widetilde{G}) = L_k(f^1|\Pi_0)$ for $k = 2, 3, 4$, and thus we only need to show that

$$L_1(\widetilde{G}) < L_1(f^1|\Pi_0). \tag{4.11}$$

To this end we will use the following lemma.

Lemma 4.4. *For all $z \in \Pi_0$ the derivative of $\widetilde{G} = \widetilde{g}^1|\Pi_0$ along E_f^{ua} has the form*

$$d\widetilde{G}_\beta(z)|E_f^{ua}(z) = \begin{pmatrix} \eta A(\beta, z) & \eta B(\beta, z) \\ C(\beta, z) & D(\beta, z) \end{pmatrix}, \tag{4.12}$$

where

$$\begin{aligned}
A = A(\beta, z) &= 1 - \beta r\sigma_r \sin\theta\cos\theta - \frac{\beta^2\sigma^2}{2} - \beta^2 r\sigma\sigma_r \cos^2\theta + O(\beta^3),\\
B = B(\beta, z) &= -\beta\sigma - \beta r\sigma_r \sin^2\theta - \beta^2 r\sigma\sigma_r \sin\theta\cos\theta + O(\beta^3),\\
C = C(\beta, z) &= \beta\sigma + \beta r\sigma_r \cos^2\theta - \beta^2 r\sigma\sigma_r \sin\theta\cos\theta + O(\beta^3),\\
D = D(\beta, z) &= 1 + \beta r\sigma_r \sin\theta\cos\theta - \frac{\beta^2\sigma^2}{2} - \beta^2 r\sigma\sigma_r \sin^2\theta + O(\beta^3).
\end{aligned}$$

Proof of the lemma. The desired relation (4.12) is apparent for $z \in \Pi_0 \backslash B$ since $\widetilde{G} = f^1$ and $\sigma = 0$ on $\Pi_0 \backslash B$. Given $z = (r, \theta, 0, s, b) \in B$ in the ua-cylindrical coordinate, by Lemma 4.3, we have that $\widetilde{g}^t(z) = (r, \theta + \theta(t), t, s, b)$ where $\theta(t) = \beta \int_0^t \widetilde{\psi}(\widetilde{g}^\tau(z))d\tau$ for $0 \le t \le 1/2$. In particular, the coordinate of $\widetilde{g}^{\frac{1}{2}}(z)$ is $(r, \theta + \beta\sigma, 1/2, s, b)$, where

$$\sigma = \sigma(r, s, b) = \frac{1}{4}\psi\left(\frac{r^2}{\varepsilon^2}\right)\psi\left(\frac{b}{\varepsilon}\right)\psi\left(\frac{s}{\varepsilon}\right)\int_{-1}^{1}\psi(u)du.$$

Back in the Cartesian coordinate system (u, a, τ, s, b), we obtain that

$$\begin{aligned}
\widetilde{g}^{\frac{1}{2}}(z) &= (u_1, a_1, 1/2, s, b)\\
&:= (u\cos(\beta\sigma) - a\sin(\beta\sigma), u\sin(\beta\sigma) + a\cos(\beta\sigma), 1/2, s, b),
\end{aligned}$$

and hence,

$$\widetilde{G}(z) = \widetilde{g}^1(z) = f^{\frac{1}{2}}\widetilde{g}^{\frac{1}{2}}(z) = (u_1, a_1, 1, s, b) = (\eta u_1, a_1, 0, \eta^{-1}s, b).$$

The last equality follows from (A.1) and the fact that $\widetilde{g}^1(z_0) = f^1(z_0)$, where z_0 is the center of B. Since $\widetilde{G}$ preserves the E_f^{ua} subbundle, we have that

$$d\widetilde{G}_\beta(z)|E_f^{ua}(z) = \begin{pmatrix} \eta A(\beta, z) & \eta B(\beta, z) \\ C(\beta, z) & D(\beta, z) \end{pmatrix},$$

where

$$A(\beta, z) = \frac{\partial u_1}{\partial u}, \quad B(\beta, z) = \frac{\partial u_1}{\partial a}, \quad C(\beta, z) = \frac{\partial a_1}{\partial u}, \quad D(\beta, z) = \frac{\partial a_1}{\partial a}.$$

Then

$$\begin{aligned} A = \frac{\partial u_1}{\partial u} &= \cos(\beta\sigma) + [-u\sin(\beta\sigma) - a\cos(\beta\sigma)]\beta\sigma_u \\ &= \cos(\beta\sigma) - \beta r \sin(\theta + \beta\sigma)\sigma_r \cos\theta \\ &= 1 - \beta r\sigma_r \sin\theta\cos\theta - \frac{\beta^2\sigma^2}{2} - \beta^2 r\sigma\sigma_r \cos^2\theta + O(\beta^3). \end{aligned}$$

Similarly, we can obtain the formulae for B, C and D. □

Lemma 4.4 allows us to follow the line of argument in the proof of Lemma 4.1 in [10] to establish (4.11). For the reader's convenience we outline the argument here.

Denote by $e_\beta(z)$ the unique number such that the vector $v_\beta(z) = (1, e_\beta(z))^t \in E^u_{\widetilde{G}_\beta}(z)$ for all $z \in \Pi_0$. One can show that

$$\begin{aligned} L_\beta = L_1(\widetilde{G}_\beta) = &\int_{\Pi_0} \log\eta \; dm(z) \\ &- \int_{\Pi_0} \log[D(\beta, z) - \eta B(\beta, z) e_\beta(\widetilde{G}_\beta(z))] dm(z). \end{aligned}$$

Note that $L_0 = L_1(f^1|\Pi_0)$, and we will show that

$$\frac{dL_\beta}{d\beta}|_{\beta=0} = 0, \quad \frac{d^2 L_\beta}{d\beta^2}|_{\beta=0} < 0, \tag{4.13}$$

which immediately implies that (4.11) holds for all sufficiently small $\beta > 0$.

To show (4.13) observe that

$$\frac{dL_\beta}{d\beta}|_{\beta=0} = -\int_{\Pi_0} D_\beta|_{\beta=0} \; dm(z) = 0$$

thus proving the first relation in (4.13). To prove the second relation note that

$$\frac{d^2 L_\beta}{d\beta^2}|_{\beta=0} = \int_{\Pi_0} [(D_\beta)^2 - D_{\beta\beta} + 2\eta B_\beta \frac{\partial}{\partial\beta}(e_\beta(\widetilde{G}_\beta(z)))]_{\beta=0} \; dm(z).$$

This integral can be written as

$$\begin{aligned} &\int_{\Pi_0} [(D_\beta(0, z))^2 - D_{\beta\beta}(0, z) + 2\eta B_\beta(0, z) C_\beta(0, z)] \; dm(z) \\ &\qquad + \int_{\Pi_0} \sum_{i=1}^{\infty} \frac{1}{\eta^i} 2B_\beta(0, z) C_\beta(0, f^{-i}(z)) \; dm(z). \end{aligned} \tag{4.14}$$

The first term in (4.14) is bounded from above by

$$-(1-\varepsilon)\int_{\Pi_0}\sigma^2 dm(z)-\frac{1}{8}\int_{\Pi_0} r^2\sigma_r^2 dm(z).$$

To estimate the second term in (4.14) note that

$$\int_{\Pi_0} 2B_\beta(0,z)C_\beta(0,f^{-i}(z))dm(z)\le 4\int_{\Pi_0}(\sigma^2+r^2\sigma_r^2)\ dm(z)$$

and that $B_\beta(0,z)C_\beta(0,f^{-i}(z))=0$ for all $z\in\Pi_0\backslash B$ and all i. Moreover, $B_\beta(0,z)C_\beta(0,f^{-i}(z))=0$ for every $z\in B$ and $i=1,\dots,N_0-1$ since $f^i(B)\cap B=\emptyset$. This allows us to take $N_0>0$ large enough to ensure that the second term is bounded by

$$\frac{1}{10}\int_B(\sigma^2+r^2\sigma_r^2)\ dm(z).$$

Hence,

$$\frac{d^2L_\beta}{d\beta^2}|_{\beta=0}\le-(\frac{9}{10}-\varepsilon)\int_{\Pi_0}\sigma^2 dm(z)-\frac{1}{40}\int_{\Pi_0} r^2\sigma_r^2 dm(z)<0.$$

This completes the proof of the inequality (4.11) thus guaranteeing that for any sufficiently small $\lambda>0$ the level set

$$\Pi=\{z\in\Pi_0:\ \lambda_1(z,\widetilde{G})\ge\lambda_2(z,\widetilde{G})>\lambda\}$$

has positive volume. It is also invariant under $\widetilde{G}$. Since $f^i(B)\cap B=\emptyset$ for $i=1,\dots,N_0$, we obtain that the sets $\widetilde{g}^i(\Pi\cap B)=\Pi\cap\widetilde{g}^i(B)=\Pi\cap f^i(B)$ corresponding to different i are pairwise disjoint subsets of Π. This implies that

$$m(\Pi)\ge N_0 m(\Pi\cap B)\ge 20k_0 m(\Pi\cap B)>0$$

thus completing the proof of Proposition 4.2. $\square$

4.2. **Construction of the flow g^t.** We perturb the flow $\widetilde{g}^t$ to a flow g^t by adding a vector field $\mathcal{X}_R$ to the vector field $\mathcal{X}_{\widetilde{g}}$. We obtain $\mathcal{X}_R$ as a sum of rotational vector fields in the ab-direction along several pairwise disjoint tubes so that the total rotation along an orbit that passes through these tubes is $\pi/2$. This ensures positive Lyapunov exponents along the E_f^{uab} subbundle for the flow g^t.

Note that there is $M_0>0$ such that for any flow F^t that is sufficiently C^1-close to the flow f^t

$$\|F^1-f^1\|_{C^1}\le M_0\|\mathcal{X}_F-\mathcal{X}_f\|_{C^1}. \tag{4.15}$$

According to Lemma B.5, M_0 depends only on the Riemannian metric and the start-up flow f^t.

Let the number $\lambda > 0$ and the subset $\Pi \subset \Pi_0$ be as in Statement (5) of Proposition 4.2. Given $K > 0$, let

$$\Lambda' = \Lambda'(K) = \{z \in \Pi : \left|\frac{1}{k}\log\|d\widetilde{g}^k(z,v)\| - \lambda\right| \le 0.1\lambda, \text{ for all } v \in E_f^{ua}(z), \|v\| = 1 \text{ and all } |k| \ge 0.5K\} \tag{4.16}$$

and

$$\Lambda = \Lambda(K) = \bigcap_{i=0}^{k_0-1} \widetilde{g}^{-i}\Lambda'(K), \tag{4.17}$$

where $k_0 > 0$ is a number which will be defined later (see Lemma 4.5). Since $m(\Lambda'(K)) \to m(\Pi)$ as $K \to \infty$ and hence, $m(\Lambda(K)) \to m(\Pi)$ as $K \to \infty$, one can choose K so large that

$$K\lambda \ge \max\{5k_0\lambda, 10\log 2, -10k_0\log(1 - M_0\delta_g)\}, \tag{4.18}$$

$$\lambda m(\Pi) + 40\log(1 - M_0\delta_g)m(\Pi\backslash\Lambda) > 0, \tag{4.19}$$

$$20m(\Pi\backslash\Lambda) \le m(\Pi). \tag{4.20}$$

Set

$$\Lambda^* = \Lambda\backslash \bigcup_{i=0}^{k_0-1} \widetilde{g}^{-i}(\mathrm{Proj}_{\Pi_0}(\Omega_0 \cup \widetilde{\Omega}_R)), \tag{4.21}$$

where Ω_0 and $\widetilde{\Omega}_R$ are given by (4.4) and (4.5) respectively. If the number ν is chosen small enough, Statement (5) of Proposition 4.2, allows us to assume that

$$\begin{aligned} m(\mathrm{Proj}_{\Pi_0}\Omega_0 \cap \Pi) &\le m(\Pi)/20k_0, \\ m(\mathrm{Proj}_{\Pi_0}\widetilde{\Omega}_R \cap \Pi) = m(B \cap \Pi) &\le m(\Pi)/20k_0. \end{aligned} \tag{4.22}$$

Combining (4.20), (4.21) and (4.22), we find that

$$m(\Lambda^*) \ge 0.8m(\Pi). \tag{4.23}$$

By Statement (5) of Proposition 4.2, the set Π is invariant under the time-1 map of the flow $\widetilde{g}^t$.

We will approximate the set Π by constructing an appropriate Rokhlin-Halmos tower (see [11]) for the map $\widetilde{g}^1$. More precisely, we choose a measurable subset $\Gamma' \subset \Pi$ such that the sets $\widetilde{g}^i(\Gamma')$ are pairwise disjoint for $-K \le i \le 6K + k_0 - 1$ and

$$m\left(\bigcup_{i=-K}^{6K+k_0-1} \widetilde{g}^i\Gamma'\right) \ge 0.9m(\Pi). \tag{4.24}$$

Consider the set Γ_0 of first entries of orbits $\{\widetilde{g}^i(z)\}_{i=0}^{5K-1}$ (with $z \in \Gamma'$) to the set Λ^*. More precisely, set

$$\Gamma_0 = \{\widetilde{g}^j(z) : z \in \Gamma',\ 0 \le j \le 5K-1,\ \widetilde{g}^j(z) \in \Lambda^*,\ \widetilde{g}^i(z) \notin \Lambda^* \text{ for } i < j\}$$

and let

$$\Gamma_i = \widetilde{g}^i(\Gamma_0), \quad \Gamma = \bigcup_{i=-K}^{K+k_0-1} \Gamma_i. \tag{4.25}$$

Clearly, the sets $\{\Gamma_i\}$ are pairwise disjoint for $-K \le i \le K + k_0 - 1$. We then approximate Γ_0 by finitely many disjoint ab-cylinders B_{0j} of the form

$$\begin{aligned} B_{0j} &= B^u(z_j, r'_j) \times B^s(z_j, r''_j) \times B^{ab}(z_j, r_j) \\ &= \{(u_j, s_j, a_j, b_j) :\ |u_j| \le r'_j, |s_j| \le r''_j, a_j^2 + b_j^2 \le r_j^2\} \\ &= \{(u_j, s_j, \rho_j, \varphi_j) :\ |u_j| \le r'_j, |s_j| \le r''_j, \rho_j \le r_j\}, \end{aligned}$$

where $r'_j, r''_j, r_j > 0$ for $j = 1, \ldots, J$ and $z_j = (u_j, s_j, a_j, b_j) = (u_j, s_j, \rho_j, \varphi_j) \in \Pi_0$ is the center of B_{0j}. For $i = -K, \ldots, K + k_0 - 1$ set

$$B_{ij} = \widetilde{g}^i(B_{0j}), \quad \Delta_i = \bigcup_{j=1}^{J} B_{ij}. \tag{4.26}$$

We can choose the sets B_{0j} in such a way that

(1) $B_{ij} \cap B_{kl} = \emptyset$ for $(i,j) \ne (k,l)$ with $-K \le i,k \le K + k_0 - 1$ and $1 \le j,l \le J$;

(2) for each $i = 0, 1, \ldots, k_0$

$$m(\Gamma_i \triangle \Delta_i) \le 0.05 \max\{m(\Gamma_i), m(\Delta_i)\}; \tag{4.27}$$

(3) $B_{ij} \cap \mathrm{Proj}_{\Pi_0}(\Omega_0 \cup \widetilde{\Omega}_R) = \emptyset$ for $0 \le i \le k_0 - 1$, $1 \le j \le J$.

The last property implies that the set $B_{ij} = \widetilde{g}^i(B_{0j}) = f^i(B_{0j})$ lies in a neighborhood around $f^i(z_j)$ and hence is still an ab-cylinder if the numbers r_j, r'_j, r''_j are chosen small enough.

We need the following lemma.

Lemma 4.5. *Given $\delta > 0$, there is $\theta_0 = \theta_0(\delta) > 0$ such that for any $\theta \in [0, \theta_0]$ and any tube $T = C \times [0, 1/2]$, where $C \subset \Pi_0$ is an ab-cylinder of the form*

$$C = B^u(z, r') \times B^s(z, r'') \times B^{ab}(z, r),$$

there exist a subtube $T' = C' \times [1/40, 19/40] \subset T$, where $C' \subset C$ is a cylinder of the form

$$C' = B^u(z, r'_0) \times B^s(z, r''_0) \times B^{ab}(z, r_0),$$

and a C^∞ vector field $\mathcal{X} = \mathcal{X}_{T,\theta}$ on $\mathcal{M}$ such that

(1) $\mathcal{X}$ is a rotation vector field with speed θ in the ab-plane, i.e.,

$$\mathcal{X}(z) = \mathcal{X}(u, s, a, b, \tau) = \theta(-b\frac{\partial}{\partial a} + a\frac{\partial}{\partial b}), \quad z \in T';$$

(2) $\mathcal{X} = 0$ outside T;
(3) $m(C')/m(C) \geq 0.75$;
(4) $r_0/r,\ r_0'/r',\ r_0''/r'' \geq 0.9$;
(5) $\|\mathcal{X}\|_{C^1} < \delta$.

Moreover, let $k_0 > 0$ be such that

$$\bar{\theta} := \frac{2\pi}{k_0 \int_{-1}^{1} \psi(t)dt} < \theta_0(\delta_g/2), \tag{4.28}$$

where $\psi(t)$ is the function in Section 4.1 and δ_g is given by Proposition 4.1. Then $\|\mathcal{X}_{T,\bar{\theta}}\| \leq \delta_g/2$.

Proof of the lemma. Given $0 < \alpha < 1$, we define a subcylinder

$$C_\alpha = B^u(z, \alpha r') \times B^s(z, \alpha r'') \times B^{ab}(z, \alpha r) \subset C.$$

By (A.2), the volume of C_α and C is induced by the flat metric $du^2 + ds^2 + da^2 + db^2$, and hence the ratio $m(C_\alpha)/m(C)$ depends only on α but not on the cylinder C. It follows that $m(C_\alpha)/m(C) \to 1$ as $\alpha \to 1$.

Fix $\alpha > 0.9$ such that $m(C_\alpha)/m(C) > 0.75$, and set $C' = C_\alpha$. Let us choose a C^∞ function $\xi : \mathbb{R} \to [0, 1]$ satisfying:

(1) $\xi = 1$ on $(-\alpha, \alpha)$;
(2) $\xi > 0$ on $(-1, 1)$ and $\xi = 0$ outside $(-1, 1)$;
(3) $\|\xi\|_{C^1} \leq \frac{2}{1-\alpha}$.

We introduce the ab-cylindrical coordinate (u, s, ρ, φ), and define a C^∞ rotational vector field $\mathcal{X} = \mathcal{X}_{T,\theta}$ by the formula

$$\mathcal{X}_{T,\theta}(z) = \begin{cases} \theta\widetilde{\xi}(z)\dfrac{\partial}{\partial\varphi}, & z \in T, \\ \\ 0, & z \in \mathcal{M}\backslash T, \end{cases} \tag{4.29}$$

where

$$\widetilde{\xi}(z) = \widetilde{\xi}(u, s, \rho, \xi, \tau) = \xi(\frac{u}{r'})\xi(\frac{s}{r''})\xi(\frac{\rho}{r})\psi(\frac{\tau - 1/4}{1/4})$$

and ψ is the smooth function in Section 4.1. Note that $\|\widetilde{\xi}\dfrac{\partial}{\partial\varphi}\| \leq c$ where $c > 0$ depends only on α but not on the choice of the cylinder C. Thus for any $\delta > 0$, there is $\theta_0 = \theta_0(\delta) > 0$ such that $\|\mathcal{X}\|_{C^1} < \delta$ for any $\theta \in [0, \theta_0]$. □

Consider the tubes $T_{ij} = B_{ij} \times [0, 1/2]$. Applying Lemma 4.5 with $T = T_{ij}$, we obtain a vector field $\mathcal{X}_{R,ij} = \mathcal{X}_{T_{ij},\bar\theta}$ such that $\|\mathcal{X}_{R,ij}\| \le \delta_g/2$, where $\bar\theta$ is given by (4.28). Moreover, there is a sub-cylinder $B'_{ij} \subset B_{ij}$ such that $m(B'_{ij})/m(B_{ij}) \ge 0.75$. Furthermore, by Lemma 4.5, we may assume that $\widetilde{g}^i(B'_{0j}) = B'_{ij}$ for $i = 1, \dots, k_0$. Finally, let

$$\Delta'_i = \bigcup_{j=1}^{J} B'_{ij}, \quad \Omega_R = \bigcup_{i=0}^{k_0-1} \bigcup_{j=1}^{J} T_{ij}, \tag{4.30}$$

and define the vector field $\mathcal{X}_R$ by

$$\mathcal{X}_R = \sum_{i=0}^{k_0-1} \sum_{j=1}^{J} \mathcal{X}_{R,ij}. \tag{4.31}$$

We obtain a new flow g^t generated by the vector field $\mathcal{X}_g = \mathcal{X}_{\widetilde{g}} + \mathcal{X}_R$. Clearly, g^t is a C^∞ volume preserving flow since $\mathcal{X}_R$ is divergence free. We will show that the flow g^t has all the desired properties stated in Proposition 4.1.

Proof of Proposition 4.1. Statements (1) and (2) follow immediately from the construction of the flow g^t and Statement (3) can be proved in the same way as Statement (3) of Proposition 4.2.

We will prove Statement (4). We need the following statement whose proof is very similar to the proof of Lemma 4.3.

Lemma 4.6. *Given $z \in B_{ij}$, the ρ_j-, u_j-, s_j- and τ- coordinates of $g^t(z)$ and $f^t(z)$ are the same for $t \in [0, 1/2]$. Consequently, $g^{\frac{1}{2}}(B_{ij}) = f^{\frac{1}{2}}(B_{ij})$ and hence Π_0 is a global cross-section for the flow $g^t|\mathcal{N} \times U_0$.*

By the lemma, the time-1 map g^1 restricted to Π_0 is the Poincaré return map of g^t on Π_0. Therefore, (4.2) is equivalent to

$$L_4(G) = 0 < L_1(G) < L_2(G) < L_3(G), \tag{4.32}$$

where $G = g^1|\Pi_0$. In fact, by (4.1) and (4.7) we have for $k = 3, 4$ that

$$L_k(G) = L_k(f^1|\Pi_0) = L_k(\widetilde{g}^1|\Pi_0) = L_k(\widetilde{G}).$$

Hence, we only need to show that

$$L_2(G) < L_3(G). \tag{4.33}$$

We follow the argument in Section 4.2 in [10] and give a sketch of the proof of (4.33).

Set $\Delta_0^* = \Delta_0' \cap \Lambda$, where Δ_0' and Λ are given by (4.30) and (4.17) respectively, and

$$\begin{aligned} U_1 &= G^{-K}\Delta_0^*, & U_2 &= \Delta_0 \backslash \Delta_0^*, \\ U_3 &= G^{k_0}((\Delta_0 \cap \Lambda)\backslash \Delta_0^*), & U_4 &= G^{k_0}(\Delta_0 \backslash \Lambda). \end{aligned}$$

Consider the first return map $\overline{G} = G^R$ on the set

$$U = U_1 \cup U_2 \cup U_3 \cup U_4 \subset \Pi_0,$$

where $R = R(z)$ is the first return time of $z \in U$ to U under G. Note that the flow g^t preserves the E_f^{uab}-subbundle, and so does G.

We intend to show that

$$\int_U \left(\log \| \wedge^3 (d\overline{G}|E_f^{uab}(z))\| - \log \| \wedge^2 (d\overline{G}|E_f^{uab}(z))\|\right) dm(z) > 0, \tag{4.34}$$

where

$$\wedge^k (d\overline{G}|E_f^{uab}(z)) : \ \wedge^k (E_f^{uab}(z)) \to \wedge^k (E_f^{uab}(z))$$

is the k-th exterior power of $d\overline{G}|E_f^{uab}(z)$. Indeed, assuming that (4.34) holds, consider the G-invariant set

$$\Pi' = \bigcup_{i=-\infty}^{\infty} G^i(U) \subset \Pi_0.$$

For $k = 2, 3$ we have that

$$\begin{aligned} \int_U \log \| \wedge^k (d\overline{G}|E_f^{uab}(z))\| dm(z) &= \int_{\Pi'} \log \| \wedge^k (dG|E_f^{uab}(z))\| dm(z) \\ &= \int_{\Pi'} \sum_{i=1}^k \lambda_i(z, G) dm(z) = L_k(G|\Pi') \end{aligned}$$

and hence, (4.34) implies that $L_2(G|\Pi') < L_3(G|\Pi')$. Since $G = \widetilde{G}$ outside Π', we obtain that $L_2(G) < L_3(G)$.

To show (4.34) we split the integral over U into four integrals I_1, I_2, I_3 and I_4 over the domains U_1, U_2, U_3 and U_4 respectively, and we obtain lower bounds for each of them. Namely, we will show that

$$\begin{aligned} I_1 &\geq 0.85K\lambda \cdot 0.7m(\Delta_0), & I_2 &\geq k_0 \log(1 - M_0\delta_g) \cdot 0.25m(\Delta_0), \\ I_3 &\geq 0, & I_4 &\geq 2\log(1 - M_0\delta_g)m(\Pi\backslash\Lambda). \end{aligned} \tag{4.35}$$

The lower bounds for I_2, I_3 and I_4 can be obtained using arguments in the proof of Lemma 4.2 in [10]. However, the proof of the lower bound for I_1 in our continuous-time case requires substantial changes and we will present it here. We need the following lemma.

Lemma 4.7. *Let $z \in U_1 = G^{-K}(\Delta_0^*)$. Then for any $v \in E_f^{uab}(z)$, we have*

$$\|d_z\overline{G}(v)\| \geq \frac{\sqrt{2}}{2}\|v\|e^{0.9K\lambda} \tag{4.36}$$

Proof of the lemma. Note that for any $z \in G^{-K}(\Delta_0^*)$, the first return time $R(z)$ is at least $2K + k_0$. Set

$$z_1 = G^K(z), \quad z_2 = G^{k_0}(z_1) = G^{K+k_0}(z), \quad z_3 = \overline{G}(z) = G^{R(z)}(z).$$

Since the orbit segments $\{g^t(z)\}_{0\leq t\leq K}$ from z to z_1 and $\{g^t(z_2)\}_{0\leq t\leq R(z)-K-k_0}$ from z_2 to z_3 are outside the set Ω_R, we have that

$$z_1 = G^K(z) = \widetilde{G}^K(z), \quad z_3 = G^{R(z)-K-k_0}(z_2) = \widetilde{G}^{R(z)-K-k_0}(z_2).$$

On the other hand, since $z_1 \in \Delta_0^* = \Delta_0' \cap \Lambda$, we can assume that $z \in B_{0j}'$ for some j, and by our construction, we have $G^i(z_1) \in B_{ij}'$ for $i = 1, \dots, k_0 - 1$, and every cylinder B_{ij}' is inside a local coordinate neighborhood of its center. Therefore, we write $z_1 = (u, s, a, b, 0) \in B_{0j}'$, and apply the similar arguments as in the proof of Lemma 4.4, we have that

$$\begin{aligned} G(z_1) = f^{\frac{1}{2}}g^{\frac{1}{2}}(z_1) &= f^{\frac{1}{2}}(u, s, a\cos\phi - b\sin\phi, a\sin\phi + b\cos\phi, 1/2) \\ &= (u, s, a\cos\phi - b\sin\phi, a\sin\phi + b\cos\phi, 1) \\ &= (\eta u, \eta^{-1}s, a\cos\phi - b\sin\phi, a\sin\phi + b\cos\phi, 0), \end{aligned}$$

where

$$\phi = \frac{1}{4}\bar{\theta}\int_{-1}^{1}\psi(t)dt = \frac{\pi}{2k_0}.$$

Repeating this calculation for $G^1(z_1), G^2(z_1), \dots, G^{k_0-1}(z_1)$ and observing that $k_0\phi = \frac{\pi}{2}$, we obtain that

$$\begin{aligned} z_2 = G^{k_0}(z_1) &= (\eta^{k_0}u, \eta^{-k_0}s, a\cos(k_0\phi) \\ &\quad - b\sin(k_0\phi), a\sin(k_0\phi) + b\cos(k_0\phi), 0) \\ &= (\eta^{k_0}u, \eta^{-k_0}s, -b, a, 0). \end{aligned}$$

This formula means that $d_{z_1}G^{k_0}$ is non-contracting along the E_f^{uab} subbundle and rotates the vector in E_f^{ab} by the angle $\pi/2$.

To obtain (4.36), we write $v = v^{ua} + v^b \in E_f^{ua}(z) \oplus E_f^b(z)$ and consider the following two cases:

(1) if $\|v^b\| \leq \frac{\sqrt{2}}{2}\|v\|$, since $d_zG^K = d_z\widetilde{G}^K$ and $z \in G^{-K}\Delta_0' \subset \widetilde{G}^{-K}\Lambda'$, by (4.16) and (4.17), we find that

$$\|d_zG^Kv\| = \|d_z\widetilde{G}^Kv\| \geq \|d_z\widetilde{G}^Kv^{ua}\| \geq \|v^{ua}\|e^{0.9K\lambda} \geq \frac{\sqrt{2}}{2}\|v\|e^{0.9K\lambda},$$

and hence

$$\|d_z\overline{G}v\| = \|d_{z^2}\widetilde{G}^{R(z)-K-k_0}d_{z_1}G^{k_0}d_z\widetilde{G}^K v\| \geq \|d_z\widetilde{G}^K v\| \geq \frac{\sqrt{2}}{2}\|v\|e^{0.9K\lambda}.$$

(2) if $\|v^b\| \geq \frac{\sqrt{2}}{2}\|v\|$, since $d_{z_1}G^{k_0}$ rotates the vector in E_f^{ab} by the angle $\pi/2$, we have

$$d_zG^{K+k_0}v^b = d_{z_1}G^{k_0}(d_z\widetilde{G}^K v^b) \in E_f^{ua}(z_2).$$

Since $z_2 \in \Lambda'$, by (4.16) we obtain

$$\begin{aligned}\|d_z\overline{G}v\| \geq \|d_z\overline{G}v^b\| &= \|d_{z_2}\widetilde{G}^{R(z)-K-k_0}d_zG^{K+k_0}v^b\| \\ &\geq \|d_zG^{K+k_0}v^b\|e^{0.9K\lambda} \geq \frac{\sqrt{2}}{2}\|v\|e^{0.9K\lambda}.\end{aligned}$$

□

By the lemma and (4.18), we find that

$$\log\|d_z\overline{G}(v)\| \geq 0.9K\lambda - 0.5\log 2 + \log\|v\| \geq 0.85K\lambda + \log\|v\|,$$

for any $z \in U_1$ and $v \in E_f^{uab}(z)$. Hence,

$$\log\|\wedge^3(d\overline{G}|E_f^{uab}(z))\| - \log\|\wedge^2(d\overline{G}|E_f^{uab}(z))\| \geq 0.85K\lambda.$$

On the other hand, it is proved in [10] that $m(U_1) \geq 0.7m(\Delta_0)$. Therefore,

$$\begin{aligned}I_1 &= \int_{U_1}\left(\log\|\wedge^3(d\overline{G}|E_f^{uab}(z))\| - \log\|\wedge^2(d\overline{G}|E_f^{uab}(z))\|\right)dm(z) \\ &\geq 0.85K\lambda\cdot 0.7m(\Delta_0).\end{aligned}$$

It follows from (4.35) that

$$\begin{aligned}&\int_U\left(\log\|\wedge^3(d\overline{G}|E_f^{uab}(z))\| - \log\|\wedge^2(d\overline{G}|E_f^{uab}(z))\|\right) \\ &\quad\geq 0.595\lambda Km(\Delta_0) + 0.25k_0\log(1-M_0\delta_g)m(\Delta_0) \\ &\quad+ 2\log(1-M_0\delta_g)m(\Pi\backslash\Lambda) \\ &\quad\geq 0.57\lambda Km(\Delta_0) + 2\log(1-M_0\delta_g)m(\Pi\backslash\Lambda) \\ &\quad\geq 0.0627\lambda m(\Pi) - 0.05\lambda m(\Pi) = 0.0127\lambda m(\Pi) > 0.\end{aligned}$$

The last two inequalities follow from (4.18), (4.19) and Sublemma 4.4 in [10] that states that $m(\Delta_0) \geq 0.11K^{-1}m(\Pi)$. This completes the proof of Statement (4) of Proposition 4.1. □

5. Accessibility

Notice that the flow g^t has positive central exponents on a set of positive volume but is not necessarily ergodic. We will perturb g^t to a flow h^t that is pointwise partially hyperbolic on the open set $\mathcal{U}$ and still has positive central exponents. Furthermore, we will ensure that the flow h^t possesses two transversal strongly stable and unstable foliations W_h^s and W_h^u of $\mathcal{U}$ and satisfies the accessibility property on $\mathcal{U}$ via these two foliations. In view of Theorem 2.4, h^t is indeed the desired flow in our Main Theorem.

We will follow the arguments in [10] and make some necessary modifications for the flow case. We choose two sequences of open subsets $U_n, \widetilde{U}_n \subset U$, $n = 1, 2, \dots$ such that

(A4) $U_0 \subset \widetilde{U}_1$;
(A5) $\widetilde{U}_n \subset \overline{\widetilde{U}_n} \subset U_n \subset \overline{U_n} \subset U$ and $\bigcup_{n\geq 1} U_n = U$;
(A6) $\widetilde{U}_n$ and U_n are connected sets for any $n \geq 1$.

Set

$$\mathcal{U}_n = \mathcal{N} \times U_n, \quad \widetilde{\mathcal{U}}_n = \mathcal{N} \times \widetilde{U}_n. \tag{5.1}$$

We will construct a sequence of flows $\{h_n^t\}_{n\geq 0}$, whose limit is the desired flow h^t. The goal of this section is to prove the following statement.

Proposition 5.1. *Given $\delta_h > 0$, one can find a sequence of positive numbers $\{\delta_n\}$ with $\delta_n \leq \min\{\delta_h/2^n, d(C, U_n)^2\}$ as well as a sequence of C^∞ divergence free vector fields $\mathcal{X}_n$ on $\mathcal{M}$, generating a sequence of volume preserving flows h_n^t, such that for $n \geq 0$*

(1) $\mathcal{X}_0 = \mathcal{X}_g$, and hence $h_0^t = g^t$;
(2) $\|\mathcal{X}_{n+1} - \mathcal{X}_n\|_{C^{n+1}} \leq \delta_n$;
(3) $\mathcal{X}_n = \mathcal{X}_f$ on $\mathcal{M}\backslash\mathcal{U}_n$, and $\mathcal{X}_n = \mathcal{X}_{n-1}$ on $\mathcal{U}_{n-2}$; in particular, each flow h_n^t is a gentle perturbation of f^t and hence satisfies Statements (3)-(5) of Proposition 3.1;
(4) for every $z \in \mathcal{M}$, we have

$$E_{h_n}^{uab\tau}(z) = E_g^{uab\tau}(z), \quad \det(dh_n^t | E_{h_n}^{uab\tau}(z)) = \det(dg^t | E_g^{uab\tau}(z));$$

(5) for all $z \in \mathcal{U}_j$, $j = 1, \dots, n$ and $\omega = u, s, c$,

$$\angle(E_{h_n}^\omega(z), E_{h_{n-1}}^\omega(z)) \leq \delta_j / 2^{n-j};$$

(6) if the number δ_g in Proposition 4.1 is sufficiently small, then each flow h_n^t is stably accessible in the following sense: Let a flow $\tilde{h}^t$ be a gentle perturbation of the flow f^t, and assume that $\angle(E_{\tilde{h}}^\omega(z), E_{h_n}^\omega(z)) \leq \delta_n$ for all $z \in \mathcal{U}_n$ and $\omega = u, s, c$. Then any

two points $z_1, z_2 \in \widetilde{\mathcal{U}}_n$ are accessible via a $(u,s)_{\tilde{h}^t}$-path in $\mathcal{U}$. In particular, h_n^t has the accessibility property in $\widetilde{\mathcal{U}}_n$.

Statements (1)–(3) imply that the limit vector field $\mathcal{X}_h = \lim_{n\to\infty} \mathcal{X}_n$ exists. Moreover,

$$\|\mathcal{X}_n - \mathcal{X}_k\|_{C^{k+1}} \le \sum_{j=k}^{n-1} \|\mathcal{X}_{j+1} - \mathcal{X}_j\|_{C^{j+1}} \le \sum_{j=k}^{n-1} \delta_j \le \delta_h/2^{k-1}$$

for any $n \ge k \ge 0$. It follows that $\mathcal{X}_n$ converges to $\mathcal{X}_h$ uniformly in the C^{k+1} topology. Since k is arbitrary, $\mathcal{X}_h$ is a C^∞ vector field. In the following section we will show that the flow h^t generated by $\mathcal{X}_h$ has all the desired properties.

5.1. **Construction of the sets U_n and $\widetilde{U}_n$.** We view the 2-torus Y as the square $[0,8]\times[0,8]$ whose opposite sides are identified. For each $n \ge 1$, consider the partition of Y into squares

$$\widehat{Z}_{ij}^{(n)} = \left[\frac{i}{2^n}, \frac{i+1}{2^n}\right] \times \left[\frac{j}{2^n}, \frac{j+1}{2^n}\right], \quad i,j = 0,1,\dots,2^{n+3}-1.$$

Without loss of generality we will assume that the square U_0, constructed in Section 3, is contained in some $\widehat{Z}_{i_0j_0}^{(1)}$ so that

$$d(U_0, \widehat{Z}_{i_0j_0}^{(1)}) \ge 1/2^4 \;\text{ and }\; d(C, \widehat{Z}_{i_0j_0}^{(1)}) > 2,$$

where C is the Cantor set constructed in Section 3. Consider the open squares

$$Z_{ij}^{(n)} = \left(\frac{i}{2^n} - \frac{1}{2^{n+2}}, \frac{i+1}{2^n} + \frac{1}{2^{n+2}}\right) \times \left(\frac{j}{2^n} - \frac{1}{2^{n+2}}, \frac{j+1}{2^n} + \frac{1}{2^{n+2}}\right),$$

$$\widetilde{Z}_{ij}^{(n)} = \left(\frac{i}{2^n} - \frac{1}{2^{n+5}}, \frac{i+1}{2^n} + \frac{1}{2^{n+5}}\right) \times \left(\frac{j}{2^n} - \frac{1}{2^{n+5}}, \frac{j+1}{2^n} + \frac{1}{2^{n+5}}\right).$$

Clearly, these squares have the same center as $\widehat{Z}_{ij}^{(n)}$ and $\widehat{Z}_{ij}^{(n)} \subset \widetilde{Z}_{ij}^{(n)} \subset Z_{ij}^{(n)}$. For $n \ge 1$ consider the set

$$Y_n = \{y \in Y :\; d(y, C) \ge 1/2^{n-2}\}.$$

Since $U_0 \subset Y_1$, we let Y_n' be the connected component of Y_n that contains U_0. Finally, consider the sets

$$\widehat{U}_1 = \widehat{Z}_{i_0j_0}^{(1)},\; U_1 = Z_{i_0j_0}^{(1)} \text{ and } \widetilde{U}_1 = \widetilde{Z}_{i_0j_0}^{(1)},$$

and for $n > 1$,

$$\widehat{U}_n = \bigcup_{\widehat{Z}_{ij}^{(n)} \cap Y_n' \neq \emptyset} \widehat{Z}_{ij}^{(n)},\;\; U_n = \bigcup_{\widehat{Z}_{ij}^{(n)} \cap Y_n' \neq \emptyset} Z_{ij}^{(n)},\;\; \widetilde{U}_n = \bigcup_{\widehat{Z}_{ij}^{(n)} \cap Y_n' \neq \emptyset} \widetilde{Z}_{ij}^{(n)}.$$

It is clear that the sets U_n and $\widetilde{U}_n$ satisfy Conditions (A4)-(A6).

Let $\widehat{\mathcal{Z}}_n = \{\widehat{Z}_{ij}^{(n)} : \widehat{Z}_{ij}^{(n)} \subset \widehat{U}_n \backslash \widehat{U}_{n-1}\}$ and $\mathcal{Z}_n = \{Z_{ij}^{(n)} : \widehat{Z}_{ij}^{(n)} \subset \widehat{\mathcal{Z}}_n\}$. Relabeling elements of $\mathcal{Z}_n$ we will denote them by $Z_1^{(n)}, \ldots, Z_{k_n}^{(n)}$, and we will use the notations $\widehat{Z}_l^{(n)}$ and $\widetilde{Z}_l^{(n)}$ for the corresponding squares contained in $Z_l^{(n)}$. Thus we have

$$U_n = U_{n-1} \cup (\bigcup_{l=1}^{k_n} Z_l^{(n)}).$$

Clearly the collection of sets $\{\widehat{Z}_l^{(n)} : n = 1, 2, \ldots, l = 1, \ldots, k_n\}$ forms a countable partition of U up to a set of zero volume while the collection of sets $\{Z_l^{(n)} : n = 1, 2, \ldots, l = 1, \ldots, k_n\}$ forms a cover of U of multiplicity at most 4. The following lemma is proved in [10].

Lemma 5.2. *There is a labeling of the squares $\{Z_l^{(n)}\}$ by integers from 1 to 8 such that for any $y \in U$, the labels of the squares $Z_l^{(n)}$ containing y are all different. In particular, $Z_1^{(1)}$ can be labeled by 1.*

5.2. **Construction of the vector fields $\mathcal{X}_n$.** The construction is similar to the one in Section 5.2 in [10], with a slightly modification on the choice of the collection of periodic points. We need the following preparations before we construct the vector fields $\mathcal{X}_n$.

Let q_j, $j = 1, \ldots, 8$ be eight periodic points of the Anosov automorphism A whose orbits are pairwise disjoint. There is $\epsilon_0 > 0$ such that

$$B_X(A^i q_j, \epsilon_0) \cap B_X(A^i q_{j'}, \epsilon_0) = \emptyset$$

whenever $j \neq j'$ and $i = -1, 0, 1$. For each q_j we choose three periodic points $p_j^i \in B_X(A^i q_j, \epsilon_0/3)$ for A, $i = a, b, \tau$, whose orbits are pairwise disjoint. Denote by $[q_j, p_j^i] = V_A^u(q_j) \cap V_A^s(p_j^i)$, $i = a, b, \tau$, where V_A^s and V_A^u are the stable and unstable local manifolds respectively. For $i = a, b, \tau$ and $j = 1, \ldots, 8$, consider the closed quadrilateral $(u, s)_A$-path γ_j^i with the collection of points q_j , $[q_j, p_j^i]$, p_j^i, $[p_j^i, q_j]$, and q_j. Without loss of generality, we will assume that $q_1 = q$, $p_1^i = p^i$ and $\gamma_1^i = \gamma^i$ for $i = a, b, \tau$ where q, p^i and γ^i are chosen as in the beginning of Section 4.1.

For $j = 1, \ldots, 8$ and $i = a, b, \tau$, we have

$$A^{\iota(q_j)}(q_j) = q_j, \qquad A^{\iota(p_j^i)}(p_j^i) = p_j^i,$$

where $\iota(q_j)$ and $\iota(p_j^i)$ are periods of q_j and p_j^i respectively. There exists $\bar{\alpha}(j, i) > 0$ such that for any $\alpha \in Y = \mathbb{T}^2$ with $\|\alpha\| \leq \bar{\alpha}(j, i)$, the

Anosov affine map $A + \alpha$ has a $\iota(q_j)$-periodic point $q_j(\alpha)$ close to q_j and a $\iota(p^i_j)$-periodic point $p^i_j(\alpha)$ close to p^i_j. Moreover, we can choose the number $\bar{\alpha}$ (in Condition (α3) at the beginning of Section 3) to be less than $\min\{\bar{\alpha}(j,i) : j = 1,\dots,8, i = a,b,\tau\}$ such that any two points from the set of periodic points

$$\{q_j(\alpha), p^i_j(\alpha) : j = 1,\dots,8, i = a,b,\tau, \|\alpha\| \le \bar{\alpha}\}$$

are disjoint.

Given $n \ge 1$ and $l = 1,\dots,k_n$, let j be the label of $Z^{(n)}_l$ in Lemma 5.2, and $y_0(n,l) = (a_0(n,l), b_0(n,l))$ the center of $Z^{(n)}_l$. We take the points associated to $Z^{(n)}_l$ as follows:

$$q(n,l) = q_j(\alpha(y_0(n,l))), \quad p^i(n,l) = p^i_j(\alpha(y_0(n,l))), \tag{5.2}$$

where $i = a,b,\tau$. Recall that η is the expanding rate of A along its unstable direction, and the function $\kappa : Y \to \mathbb{R}$ is given in the beginning of Section 3. For $n \ge 1$ let us choose a square $Z^{(n)}_l \in \mathcal{Z}_n$. In the case $n > 1$, we write for simplicity $q = q(n,l)$ and $p^i = p^i(n,l)$ and we let $\eta_-(n,l) = \min\{\eta^{\kappa(y)} : y \in Z^{(n)}_l\}$. Define the numbers

$$\begin{aligned} \alpha^i_u &= \alpha^i_u(n,l) = d(p^i, [p^i, q]), \\ \alpha^i_s &= \alpha^i_s(n,l) = d(p^i, [q, p^i]), \\ \breve{\alpha}^i_u &= \breve{\alpha}^i_u(n,l) = \alpha^i_u(n,l)/\eta_-(n,l), \\ \breve{\alpha}^i_s &= \breve{\alpha}^i_s(n,l) = \alpha^i_s(n,l)/\eta_-(n,l) \end{aligned} \tag{5.3}$$

and the rectangles in X

$$\begin{aligned} \Pi^i(n,l) &= B_{F^u}(p^i, \alpha^i_u) \times B_{F^s}(p^i, \alpha^i_s), \\ \breve{\Pi}^i(n,l) &= B_{F^u}(p^i, \breve{\alpha}^i_u) \times B_{F^s}(p^i, \breve{\alpha}^i_s). \end{aligned}$$

We will assume that the rectangles $\Pi^i(n,l)$, $n \ge 1$, $l = 1,\dots,k_n$ and $i = a,b,\tau$ are pairwise disjoint if the number $\bar{\alpha}$ is chosen sufficiently small. Finally, we let

$$\begin{aligned} \epsilon_\tau &= \epsilon_\tau(n,l) = \min\{\kappa(y)/2 : y \in Z^{(n)}_l\}, \\ \breve{\epsilon}_\tau &= \breve{\epsilon}_\tau(n,l) = 5\epsilon_\tau(n,l)/6. \end{aligned} \tag{5.4}$$

In the case $n = 1$, we have $Z^{(1)}_1 = U_1$ and $q(1,1) = q_1$, $p^i(1,1) = p^i_1$ since the function $\alpha = 0$ on U_1. Choose l^i_u and l^i_s such that

$$A^{-l^i_u}([p^i_1, q_1]) \in B_X(p^i_1, \nu/2), \quad A^{l^i_s}([q_1, p^i_1]) \in B_X(p^i_1, \nu/2),$$

where ν is given in (4.4). Then we set

$$\alpha^i_u = \alpha^i_u(1,1) = d(p^i_1, A^{-l^i_u}[p^i_1, q_1]), \quad \alpha^i_s = \alpha^i_s(1,1) = d(p^i_1, A^{l^i_s}[q_1, p^i_1])$$

with other quantities and sets to be defined in a similar way.

In addition to the squares $\widehat{Z}_{ij}^{(n)}$, $\widetilde{Z}_{ij}^{(n)}$ and $Z_{ij}^{(n)}$ constructed in the previous subsection, we need to consider the following squares:

$$\breve{Z}_{ij}^{(n)} = \left(\frac{i}{2^n} - \frac{1}{2^{n+3}}, \frac{i+1}{2^n} + \frac{1}{2^{n+3}}\right) \times \left(\frac{j}{2^n} - \frac{1}{2^{n+3}}, \frac{j+1}{2^n} + \frac{1}{2^{n+3}}\right);$$

$$\bar{Z}_{ij}^{(n)} = \left(\frac{i}{2^n} - \frac{1}{2^{n+4}}, \frac{i+1}{2^n} + \frac{1}{2^{n+4}}\right) \times \left(\frac{j}{2^n} - \frac{1}{2^{n+4}}, \frac{j+1}{2^n} + \frac{1}{2^{n+4}}\right)$$

as well as the following intervals:

$$I_n = J_n = \left(-\frac{3}{2^{n+2}}, \frac{3}{2^{n+2}}\right), \quad \breve{I}_n = \breve{J}_n = \left(-\frac{5}{2^{n+3}}, \frac{5}{2^{n+3}}\right),$$

and

$$K = (-1/4, 1+1/4), \quad \breve{K} = (-1/8, 1+1/8), \quad \bar{K} = (-1/16, 1+1/16).$$

Note that we have that

$$\widehat{Z}_{ij}^{(n)} \subset \widetilde{Z}_{ij}^{(n)} \subset \bar{Z}_{ij}^{(n)} \subset \breve{Z}_{ij}^{(n)} \subset Z_{ij}^{(n)}$$

and similar relations for I_n and J_n.

Fix $n \geq 1$ and $l = 1, \dots, k_n$, and write $\alpha_\omega^i = \alpha_\omega^i(n,l)$, $\breve{\alpha}_\omega^i = \breve{\alpha}_\omega^i(n,l)$ for $i = a, b, \tau$, $\omega = u, s$, and $\epsilon_\tau = \epsilon_\tau(n,l)$, $\breve{\epsilon}_\tau = \breve{\epsilon}_\tau(n,l)$. We choose functions as follows:

(1) ϕ^i and ψ^i are C^∞ functions on $\mathbb{R}$ such that
- ϕ^i =const. on $(-\breve{\alpha}_u^i, \breve{\alpha}_u^i)$ and ψ^i =const. on $(-\breve{\alpha}_s^i, \breve{\alpha}_s^i)$;
- $\phi^i(r) = 0$ for $|r| \geq \alpha_u^i$, $\psi^i(r) = 0$ for $|r| \geq \alpha_s^i$;
- $\int_0^{\pm\alpha_u^i} \phi^i(r)dr = 0$, and $\psi^i(r) > 0$ for any $|r| < \alpha_s^i$;
- $\|\phi^i\|_{C^n}, \|\psi^i\|_{C^n} \leq 1$.

(2) ξ_τ and ξ_Y are C^∞ functions supported on K and I_n respectively such that
- ξ_τ =const. on $\breve{K}$, and ξ_Y =const. on $\breve{I}_n$;
- $\xi_\tau(r) > 0$ for $r \in K$, and $\xi_Y(r) > 0$ for $r \in I_n$;
- $\xi_\tau(r) = 0$ for $r \notin K$, and $\xi_Y(r) = 0$ for $r \notin I_n$;
- $\|\xi_\tau\|_{C^n}, \|\xi_Y\|_{C^n} \leq 1$.

(3) ζ_τ and ζ_Y are C^∞ functions supported on $(-\epsilon_\tau, \epsilon_\tau)$ and I_n respectively such that
- ζ_τ =const. on $(-\breve{\epsilon}_\tau, \breve{\epsilon}_\tau)$, and ζ_Y =const. on $\breve{I}_n$;
- $\zeta_\tau(r) > 0$ for $r \in (-\epsilon_\tau, \epsilon_\tau)$, and $\zeta_Y(r) > 0$ for $r \in I_n$;
- $\zeta_\tau(r) = 0$ for $r \notin (-\epsilon_\tau, \epsilon_\tau)$, and $\zeta_Y(r) = 0$ for $r \notin I_n$;
- $\|\zeta_\tau\|_{C^n}, \|\zeta_Y\|_{C^n} \leq 1$.

Now we are ready to construct the sequence of vector fields $\mathcal{X}_n$. Given $n \geq 1$, $l = 1, \ldots, k_n$ and $i = a, b, \tau$, take the Cartesian coordinate system $z = (u, s, \tau, a, b) = (x, \tau, a, b)$ with the origin at $(p^i(n,l), 1/2, y_0(n,l))$. In this coordinate system the interval K is in the symmetric form $(-3/4, 3/4)$. Take the boxes for $i = a, b$

$$\Omega^i = \Omega^i_{n,l} = \{(x, \tau, y) : \ x \in \Pi^i(n,l), \ |\tau| \leq \epsilon_\tau(n,l), \ y \in Z_l^{(n)}\},$$

and

$$\Omega^\tau = \Omega^\tau_{n,l} = \{(x, \tau, y) : \ x \in \Pi^\tau(n,l), \ \tau \in K, \ y \in Z_l^{(n)}\}.$$

By the construction of the rectangles $\Pi^i(n,l)$, we have that $\Omega^i(n,l) \cap \Omega^{i'}(n',l') = \emptyset$ if $(i,n,l) \neq (i',n',l')$. Similarly, we can choose $\breve{\Omega}^i$, $i = a, b, \tau$ by taking $\breve{\Pi}^i$, $\breve{\epsilon}_\tau$, $\breve{K}$ and $\breve{Z}_l^{(n)}$. Next we define three divergence free vector fields

$$\mathcal{X}^a = \mathcal{X}^a_{n,l} = \zeta_Y(b)\zeta_\tau(\tau)\psi^a(s)\Big(-\xi_Y'(a)\int_0^u \phi^a(r)dr, \ 0, \ 0, \xi_Y(a)\phi^a(u), \ 0\Big),$$

$$\mathcal{X}^b = \mathcal{X}^b_{n,l} = \zeta_Y(a)\zeta_\tau(\tau)\psi^b(s)\Big(-\xi_Y'(b)\int_0^u \phi^b(r)dr, \ 0, \ 0, \ 0, \ \xi_Y(b)\phi^b(u)\Big),$$

$$\mathcal{X}^\tau = \mathcal{X}^\tau_{n,l} = \zeta_Y(a)\zeta_Y(b)\psi^\tau(s)\Big(-\xi_\tau'(\tau)\int_0^u \phi^\tau(r)dr, \ 0, \ \xi_\tau(\tau)\phi^\tau(u), \ 0, \ 0\Big).$$

Clearly each vector field $\mathcal{X}^i_{n,l}$ vanishes outside the corresponding box $\Omega^i_{n,l}$, and it is constant on the smaller box $\breve{\Omega}^i_{n,l}$. Finally, we set

$$\widehat{\mathcal{X}}_n = \sum_{l=1}^{k_n}(\mathcal{X}^a_{n,l} + \mathcal{X}^b_{n,l} + \mathcal{X}^\tau_{n,l}), \quad \mathcal{X}_n = \mathcal{X}_g + \sum_{k=1}^{n}\beta_k\widehat{\mathcal{X}}_k, \tag{5.5}$$

where the sequence of small positive numbers $\{\beta_n\}$ is determined inductively to ensure Statements (2) and (5) of Proposition 5.1. Let h_n^t be the flow on $\mathcal{M}$ generated by the vector fields $\mathcal{X}_n$.

5.3. **Proof of Proposition 5.1.** Statements (1)-(4) follow directly from our construction. It remains to show how to choose the sequence of positive numbers δ_n such that h_n^t satisfies Statements (5) and (6) of the proposition. Note that these two statements only concern those invariant subbundles E^ω and foliations W^ω, $\omega = u, s, c, cs, cu$, which are the same for the flow and its time-1 map. Therefore, the choice of δ_n and related arguments are similar to the diffeomorphism case in [10]. We will outline the proof here.

For any gentle perturbation $h_\natural^t$ of f^t (see Definition 2.1), we denote by $W^c_{h_\natural}(z)$ the center manifold of $h_\natural^t$ at the point $z \in \mathcal{M}$. Given a square $Z_l^{(n)}$ with the center $y_0(n,l)$, let $q(n,l)$, $p^i(n,l)$, $i = a, b, \tau$

be the associated periodic points given by (5.2), and $z_0 = z_0(n,l) = (q(n,l), 1/2, y_0(n,l))$. We denote by $W^c_{h_\natural}(z_0, K, Z^{(n)}_l)$ the connected component of $W^c_{h_\natural}(z_0) \cap (X \times K \times Z^{(n)}_l)$ that contains z_0. We will also use similar notations $W^c_{h_\natural}(z_0, \breve{K}, \breve{Z}^{(n)}_l)$, etc.

Next we will introduce two important families of maps Θ and Ψ for a gentle perturbation $h^t_\natural$ of f^t.

Fix $n \geq 1$ and $l = 1, \dots, k_n$, we take the collection of points $q = q(n,l)$, $p^i = p^i(n,l)$, $i = a, b, \tau$. Consider a quadrilateral $(u,s)_{h^t_\natural}$-path $\widehat{\gamma}^i = \{z_1, \dots, z_5\}$ with initial point z_1 defined by

$$
\begin{aligned}
z_2 &= V^u_{h_\natural}(z_1) \cap V^{cs}_{h_\natural}(p^i, 1/2, a_0, b_0), \\
z_3 &= V^s_{h_\natural}(z_2) \cap V^{cu}_{h_\natural}(p^i, 1/2, a_0, b_0), \\
z_4 &= V^u_{h_\natural}(z_3) \cap V^{cs}_{h_\natural}(z_1), \\
z_5 &= V^s_{h_\natural}(z_4) \cap V^{cu}_{h_\natural}(z_1).
\end{aligned}
\tag{5.6}
$$

This path defines a map $\Theta^i = \Theta^i_{n,l,h_\natural}$ given by $\Theta^i(z_1) = z_5$. It is easy to see that $z_5 \in V^c_{h_\natural}(z_1)$, and Θ^i maps $W^c_{h_\natural}(z_0, K, Z^{(n)}_l)$ into itself. Reparameterizing the curve on $V^u_{h_\natural}(z_1)$ from z_1 to z_2 by $\sigma : [0,1] \to V^u_{h_\natural}(z_1)$ so that $\sigma(0) = z_1$ and $\sigma(1) = z_2$, we obtain a parameterized family of quadrilaterals $\widehat{\gamma}^i(\vartheta) = \{z_1(\vartheta), \dots, z_5(\vartheta)\}$, $\vartheta \in [0,1]$, where $z_1(\vartheta) = z_1$, $z_2(\vartheta) = \sigma(\vartheta)$, and $z_k(\vartheta)$, $k = 3,4,5$ are obtained in the way similar to (5.6). Then we define $\Theta^i_\vartheta = \Theta^i_{\vartheta,n,l,h_\natural}$ given by $\Theta^i_\vartheta(z_1) = z_5(\vartheta)$. Clearly $\Theta^i_0 = Id$, $\Theta^i_1 = \Theta^i$, Θ^i_ϑ maps $W^c_{h_\natural}(z_0, K, Z^{(n)}_l)$ into $W^c_{h_\natural}(z_0)$ and depends continuously on $\vartheta \in [0,1]$.

On the other hand, given $z = ((x,\tau), y) \in \mathcal{U}$, there is a $(u,s)_{f^t}$-path $\gamma_f(z)$ connecting z to $z' = ((q,\tau), y)$ whose length does not exceed $2d(x,q)$. This generates a map $\Psi_f = \Psi_{f,n,l}$ from $\mathcal{U}$ to $\{q\} \times K \times G$ given by $\Psi_f(z) = z'$. Furthermore, given a gentle perturbation $h^t_\natural$ of f^t and a point $z \in Z^{(n)}_l$, we can find a $(u,s)_{h_\natural}$-path $\gamma_{h_\natural}(z)$, which is close to $\gamma_f(z)$ and connects z to a point $z' = z'(h^t_\natural) \in W^c_{h_\natural}(z_0, K, Z^{(n)}_l)$. We can then define $\Psi_{h_\natural} = \Psi_{h_\natural,n,l}$ by $\Psi_{h_\natural}(z) = z'(h^t_\natural)$.

Note that the maps $\Psi_{h_\natural,n,l}$, $\Theta^i_{n,l,h_\natural}$ and $\Theta^i_{\vartheta,n,l,h_\natural}$, $i = a, b, \tau$ depend continuously on $h^t_\natural$ as long as $h^t_\natural$ is a gentle perturbation of f^t with $h^t_\natural = f^t$ outside some fixed $\mathcal{U}_n$ and with $\angle(E^\omega_{h_\natural}(z), E^\omega_f(z))$ sufficiently small for all $z \in \mathcal{U}_n$ and $\omega = u, s, c$. Moreover, the continuity is uniform with respect to z.

Given a set $\Gamma \subset \mathcal{M}$ and a gentle perturbation $h_\natural^t$ of f^t, set

$$\mathcal{A}_{h_\natural}(\Gamma) = \{z \in \mathcal{M} : \text{there exists } y \in \Gamma \text{ such that } y \text{ is accessible to } z \text{ via a } (u,s)_{h_\natural^t}\text{-path}\}.$$

For $n \geq 1$ denote by $\epsilon_n = \min\{1/2^{n+5}, \breve{\epsilon}_\tau(n,l), l = 1, \dots, k_n\}$, where $\breve{\epsilon}_\tau(n,l)$ is defined by (5.4).

We now briefly describe how to choose the sequence $\{\delta_n\}$. See [10] for more details. Recall that $U_1 = Z_1^{(1)}$, $\widetilde{U}_1 = \widetilde{Z}_1^{(1)}$, and $\mathcal{U}_1 = \mathcal{N} \times U_1$, $\widetilde{\mathcal{U}}_1 = \mathcal{N} \times \widetilde{U}_1$. Choose $\theta_0 > 0$ such that the families of maps $\Psi_{h_\natural}$ and $\Theta^i_{h_\natural}$ are well-defined for any gentle perturbation $h_\natural^t$ of f^t with $\angle(E^\omega_{h_\natural}(z), E^\omega_f(z)) \leq 2\theta_0$ for $\omega = u, s, c$. We assume that the number δ_g in Proposition 4.1 is so small that the flow $h_0^t = g^t$ satisfies $\angle(E^\omega_{h_0}(z), E^\omega_f(z)) \leq \theta_0$ and $d(\Theta^i_{\vartheta,1,1,h_0}(z), z) \leq \epsilon_1/4$ for $z \in \mathcal{N} \times U_0$, $\vartheta \in [0,1]$ and $i = a, b, \tau$.

Now choose θ_1' with $0 < \theta_1' \leq \theta_0/2$ such that $d(\Psi_{h_\natural}(z), \Psi_{h_0}(z)) \leq 1/2^8$ if $\angle(E^\omega_{h_\natural}(z), E^\omega_{h_0}(z)) \leq 2\theta_1'$ for all $z \in \mathcal{N} \times Z_1^{(1)}$. Also choose $\delta_1' > 0$ such that if $\|\mathcal{X}_1 - \mathcal{X}_0\| \leq \delta_1'$, then $\angle(E^\omega_{h_1}(z), E^\omega_{h_0}(z)) \leq \theta_1'$. Finally set $\theta_1 = \min\{\theta_1', \theta_1''\}$ and $\delta_1 = \min\{\delta_1', \delta_1'', \theta_1\}$, where δ_1'' and θ_1'' are given by Lemma 5.3 below. We can show

(1) $d(\Psi_{h_1}(z), \Psi_{h_0}(z)) \leq 1/2^8$ for all $z \in \mathcal{N} \times Z_1^{(1)}$;
(2) $d(\Theta^i_{\vartheta,2,l,h_1}(z), z) \leq \epsilon_2/4$ for all $z \in W^c_{h_1}(z_0(2,l), K, Z_l^{(2)})$, $i = a, b, \tau$, $\vartheta \in [0,1]$ and $l = 1, \dots, k_2$;
(3) $\mathcal{A}_{h_\natural}(z_0(1,1)) \supset W^c_{h_\natural}(z_0(1,1), \bar{K}, \bar{Z}_1^{(1)})$ for any gentle perturbation $h_\natural^t$ of f^t, close to h_1^t, with $\angle(E^\omega_{h_\natural}(z), E^\omega_{h_1}(z)) \leq \theta_1'$, $\omega = u, s, c$ and $z \in \mathcal{N} \times Z_1^{(1)}$.

Moreover, the above statements imply that $\mathcal{A}_{h_\natural}(z_0(1,1)) \supset \mathcal{N} \times \widetilde{Z}_1^{(1)}$, in particular, h_1^t has the accessibility property on $\mathcal{N} \times \widetilde{Z}_1^{(1)}$.

Proceeding inductively, we can choose δ_n such that Statements (5) and (6) of Proposition 5.1 hold. Furthermore, we have for $i = a, b, \tau$, $\vartheta \in [0,1]$ and $l = 1, \dots, k_{n+1}$,

(1) $d(\Psi_{h_n}(z), \Psi_{h_{n-1}}(z)) \leq 1/2^{n+7}$ for all $z \in \mathcal{N} \times Z_l^{(n)}$;
(2) $d(\Theta^i_{\vartheta,n+1,l,h_n}(z), z) \leq \epsilon_{n+1}/4$ for all $z \in W^c_{h_n}(z_0(n+1,l), K, Z_l^{(n+1)})$;
(3) $\mathcal{A}_{h_\natural}(z_0(n,l)) \supset W^c_{h_\natural}(z_0(n,l), \bar{K}, \bar{Z}_l^{(n)})$ for any gentle perturbation $h_\natural^t$ of f^t, close to h_n^t, with $\angle(E^\omega_{h_\natural}(z), E^\omega_{h_n}(z)) \leq \delta_n$, $\omega = u, s, c$ and $z \in \mathcal{N} \times Z_l^{(n)}$.

Therefore, $\mathcal{A}_{h_\natural}(z_0(n,l) \supset \mathcal{N} \times \widetilde{Z}_l^{(n)}$ for all $l = 1, \dots, k_{n+1}$. In other words, $h_\natural^t$ has the accessibility property on $\mathcal{N} \times \widetilde{Z}_l^{(n)}$.

Note that

$$\widetilde{\mathcal{U}}_n = \widetilde{\mathcal{U}}_{n-1} \bigcup \left(\bigcup_{l=1}^{k_n} \mathcal{N} \times \widetilde{Z}_l^{(n)} \right),$$

and the intersection of any two sets among $\widetilde{\mathcal{U}}_{n-1}$ and $\mathcal{N} \times \widetilde{Z}_l^{(n)}$, $l = 1, \ldots, k_n$ contains a nonempty open set whenever they intersect. Since $\widetilde{\mathcal{U}}_n$ is connected, we obtain accessibility of $h_\natural^t$ on $\widetilde{\mathcal{U}}_n$. In particular, h_n^t has the accessibility property on $\widetilde{\mathcal{U}}_n$ when we apply that $h_\natural^t = h_n^t$.

5.4. **A technical lemma.** The proof of Proposition 5.1 heavily relies on the following technical statements.

Lemma 5.3. *Suppose for some $n > 0$, $d(\Theta^i_{\vartheta,n,l,h_{n-1}}(z), z) \le \epsilon_n/4$ for all $i = a, b, \tau$, $\vartheta \in [0,1]$, $z \in W^c_{h_{n-1}}(z_0(n,l), K, Z_l^{(n)})$, $l = 1, \ldots, k_n$. Then there are δ''_n, $\theta''_n > 0$ such that*

(1) if $\|\mathcal{X}_n - \mathcal{X}_{n-1}\|_{C^n} \le \delta''_n$, then we have

$$d(\Theta^i_{\vartheta,n+1,l,h_n}(z), z) \le \epsilon_{n+1}/4, \text{ as } z \in W^c_{h_n}(z_0(n+1), K, Z_l^{(n+1)}), \tag{5.7}$$

for all $i = a, b, \tau$, $\vartheta \in [0,1]$, and $l = 1, \ldots, k_{n+1}$;

(2) for any gentle perturbation $h_\natural^t$ of f^t, close to h_n^t and with

$$\angle(E^\omega_{h_\natural}(z), E^\omega_{h_n}(z)) \le \theta''_n, \text{ for all } z \in \mathcal{N} \times Z_l^{(n)}, \ \omega = u, s, c$$

we have

$$\mathcal{A}_{h_\natural}(z_0(n,l)) \supset W^c_{h_\natural}(z_0(n,l), \bar{K}, \bar{Z}_l^{(n)}) \text{ for all } l = 1, \ldots, k_n. \tag{5.8}$$

In particular, (5.8) holds with $h_\natural^t = h_n^t$.

This lemma is an adaptation of Lemma 5.2 in [10] to the flow case. It can be proved in a similar fashion subject to the following sublemma.

Sublemma 5.4. *For each $n > 0$, there exists $\delta''_n > 0$ such that if $\|\mathcal{X}_n - \mathcal{X}_{n-1}\|_{C^n} = \beta_n \|\widehat{\mathcal{X}}_n\|_{C^n} \le \delta''_n$, then for all $l = 1, \ldots, k_n$, we have*

1. *$\Theta^a((q, 1/2, a, 0)) = (q, 1/2, a', 0)$ with $a' < a$ for any $a \in I_n$;*
2. *$\Theta^b((q, 1/2, a, b)) = (q, 1/2, a, b')$ with $b' < b$ for any $a \in I_n$, $b \in J_n$;*
3. *$\Theta^\tau((q, \tau, a, b)) = (q, \tau', a, b)$ with $\tau' < \tau$ for any $a \in I_n$, $b \in J_n$ and $\tau \in K$,*

where $q = q(n,l)$ is given by (5.2), and $\Theta^i = \Theta^i_{n,l,h_n}$ for $i = a, b, \tau$.

Proof. The proof is similar to that in [6] (see Lemma B.4; see also [10], Sublemma 5.3) but is adapted to the language of vector fields.

We will prove the first statement. Consider the coordinate system (u, s, τ, a, b) in $\Omega^a_{n,l}$ with the origin at $(p^a(n, l), 1/2, y_0(n, l))$. Write $q = q(n, l)$, $p^a = p^a(n, l)$ and $y_0 = y_0(n, l)$. We may assume that the square $Z^{(n)}_l$ is parameterized as $(a, b) \in I_n \times J_n$, that the center $y_0(n, l)$ of $Z^{(n)}_l$ is $(0, 0)$ and that the local coordinates of the points q, $[q, p^a]$, $[p^a, q]$ and p^a are (u_0, s_0), $(0, s_0)$, $(u_0, 0)$ and $(0, 0)$ respectively, where $u_0 = \alpha^a_u(n, l)$ and $s_0 = \alpha^a_s(n, l)$ given by (5.3).

Consider the case $n > 1$ first. Note that the vector field $\mathcal{X}_n$ inside $\Omega^a(n, l)$ is exactly $\mathcal{X}_f + \beta_n \mathcal{X}^a_{n,l}$. For any $a_1 = a \in I_n$, $b \in J_n$, and $\tau \in (1/2 - \epsilon_\tau, 1/2 + \epsilon_\tau)$, choose the point $\underline{z}_1 = (q, \tau, a_1, b) = (u_0, s_0, \tau, a_1, b)$. Note that under the original flow f^t, we have a closed quadrilateral $(u, s)_{f^t}$-path $\underline{\gamma} = \{\underline{z}_1, \underline{z}_2, \underline{z}_3, \underline{z}_4, \underline{z}_5\}$, where

$$\begin{aligned}
\underline{z}_2 &= ([q, p^a], \tau, \underline{a}_2, b) = (0, s_0, \tau, \underline{a}_2, b),\\
\underline{z}_3 &= (p^a, \tau, \underline{a}_3, b) = (0, 0, \tau, \underline{a}_3, b),\\
\underline{z}_4 &= ([p^a, q], \tau, \underline{a}_4, b) = (u_0, 0, \tau, \underline{a}_4, b),\\
\underline{z}_5 &= ([q, p^a], \tau, \underline{a}_5, b) = (u_0, s_0, \tau, \underline{a}_5, b) = \underline{z}_1,
\end{aligned}$$

and $\underline{a}_k = a_1 = a$ for $k = 1, 2, 3, 4, 5$.

Let us compare the vector field $\mathcal{X}_n = \mathcal{X}_f + \beta_n \mathcal{X}^a_{n,l}$ on each leg $\mathcal{L}_k = [\underline{z}_k, \underline{z}_{k+1}]$ for $k = 1, 2, 3, 4$. In fact, $\mathcal{X}^a_{n,l} \equiv 0$ on legs $\mathcal{L}_1$ and $\mathcal{L}_4$. Since the u-component of every point on the leg $\mathcal{L}_2$ is 0, the u-component of the vector field $\mathcal{X}^a_{n,l}$ is 0, and the a-component does not depend on the u-coordinate. On the leg $\mathcal{L}_3 = [\underline{z}_3, \underline{z}_4]$, the u-component of $\mathcal{X}^a_{n,l}$ is negative at the interior points and it is zero at two endpoints $\underline{z}_3$ and $\underline{z}_4$, while the a-component is positive, with the value smoothly changing from a constant to zero.

Now choose the point $z_1 = \underline{z}_1$, and we have the quadrilateral $(u, s)_{h^t_n}$-path $\gamma = \{z_1, z_2, z_3, z_4, z_5\}$. By the above comparison, the τ- and b-coordinate are the same for each z_k, $k = 1, 2, 3, 4, 5$. By the construction of the vector fields $\mathcal{X}_n$, the image of the leg $[z_3, z_4]$ under the flow h^t_n is contained in Ω^a. Now let a_k be the a-coordinate of z_k, $k = 1, 2, 3, 4, 5$, since the a-component of $\mathcal{X}_n$ are the same for all points on $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_4$, while it changes from a constant to zero along the unstable leg $\mathcal{L}_3$, then we have $a_1 = a_2 = a_3 > a_4 = a_5$. This shows Statement (1) for the case $n > 1$.

In the case $n = 1$ similar arguments can be used with the following modification, and we will obtain $a_1 = a_2 \geq a_3 > a_4 = a_5$. This completes the proof of Statement (1). Statements (2) and (3) can be proved in a similar way. □

6. Proof of Main Theorem.

Since each $\mathcal{X}_n$ is divergence free, so is $\mathcal{X}_h$, and hence h^t is volume preserving. The first statement of the Main Theorem follows.

Note that $h^t = f^t$ on $\mathcal{U}^c$ and is of the form

$$h^t((x,\tau),y) = ((x+t\alpha_0,\tau),y)$$

for each $z = ((x,\tau),y) \in \mathcal{U}^c = \mathcal{N} \times C$ where α_0 is a diophantine vector (see Section 3). Hence, h^t preserves each 3-dimensional submanifold $\mathcal{N} \times \{y\}$, $y \in C$, and $h^t|\mathcal{N} \times \{y\}$ is a non-identity linear flow since $\alpha(y) \neq 0$. Moreover, the frequency vector $\alpha(y)$ is Diophantine if $y \in C$. Thus Statements (1) and (3) of the Main Theorem follows.

It remains to prove the second statement. By Proposition 4.1, each diffeomorphism h_n^t is pointwise partially hyperbolic on $\mathcal{U}$ and uniformly partially hyperbolic on $\overline{\mathcal{U}}_n$. By Theorem B.1 in the Appendix B, if the sequence δ_n decreases sufficiently fast, the limit flow h^t is pointwise partially hyperbolic on $\mathcal{U}$.

We now claim that the one-dimensional strongly stable E_h^s and unstable E_h^u subbundles are integrable to invariant strongly stable W_h^s and unstable W_h^u foliations with smooth leaves, which are transversal and absolutely continuous. Recall that the "start-up" flow f^t has strongly stable and unstable local manifolds $V_f^s(z)$ and $V_f^u(z)$ respectively at each $z \in \mathcal{U}$. Moreover, these local manifolds are of uniform size, say larger than a certain number $4r > 0$. By Proposition 5.1(3), $h_n^t|\mathcal{U}_n^c = f^t|\mathcal{U}_n^c$, and thus $V_{h_n}^\omega(z) = V_f^\omega(z)$ for all $z \in \mathcal{U}\backslash\mathcal{U}_n$, $\omega = s, u$. On the other hand, each h_n^t is a perturbation of h_{n-1}^t on the compact set $\mathcal{U}_n$, on which both h_n^t and h_{n-1}^t are uniformly partially hyperbolic if δ_n is sufficiently small. Furthermore, let r_n be the size of $V_{h_n}^\omega(z)$ for $z \in \mathcal{U}_n$, one can have $r_n/r_{n-1} \geq 2^{-1/2^n}$, and thus by induction we have that the size of local manifolds of $h_n^t|\mathcal{U}_n$ is bigger than r. Therefore, given $z \in \mathcal{U}$, we obtain that the size of $V_{h_n}^\omega(z)$ has a lower bound $r > 0$, which is independent of z and n.

Write each $V_{h_n}^s(z)$ in the coordinate chart as follows

$$V_{h_n}^s(z) = \exp_z\{(v, \psi_{h_n}^s(v)) : \ v \in B^s(0, r_n)\},$$

where $B^s(0,r_n) \subset E_{h_n}^s(z)$ is the ball centered at origin of radius r_n and $\psi_{h_n}^s : B^s(0,r_n) \to E_{h_n}^{cu}(z)$ is a C^1 map satisfying:

(1) $\psi_{h_n}^s(0) = 0$ and $d\psi_{h_n}^s(0) = 0$;

(2) If the numbers δ_n and θ_n decay sufficiently fast then there are $r > 0$ and $\Delta > 0$ such that $r_n \geq r$ and $\|\psi^s_{h_n}\|_{C^1} \leq \Delta$ for all $n \geq 0$.

This implies that $z \in V^s_{h_n}(z)$ and $T_z V^s_{h_n}(z) = E^s_{h_n}(z)$. Furthermore,

(1) $h^t_n(V^s_{h_n}(z)) \subset V^s_{h_n}(h^t_n(z))$;
(2) $d(h^t_n(z), h^t_n(y)) \leq \tilde{\lambda}(z) d(z,y)$ for each $y \in V^s_{h_n}(z)$ and some continuous function $\tilde{\lambda}(z)$ on $\mathcal{U}$ for which $0 < \lambda(z) \leq \tilde{\lambda}(z) < \lambda'(z)$ (where $\lambda'(z)$ is the function in the definition of pointwise partial hyperbolicity).

The sequence of functions $\psi^s_{h_n}(v)$, $\|v\| \leq r$, is compact in the C^1 topology and hence, there is a subsequence $\psi^s_{h_{n_k}}$ that converges to a C^1 function ψ satisfying $\psi(0) = 0$, $d\psi(0) = 0$ and $\|\psi\|_{C^1} \leq \Delta$. Setting

$$V(z) = \exp_z\{(v, \psi(v)) : \ v \in B^s(0,r)\} \tag{6.1}$$

we have that

(1) $z \in V(z)$ and $T_z V(z) = E^s_h(z)$;
(2) $h^t(V(z)) \subset V(h^t(z))$;
(3) $d(h^t(z), h^t(y)) \leq \tilde{\lambda}(z) d(z,y)$ for each $y \in V(z)$.

This implies that if m_k is any subsequence for which $\psi^s_{h_{m_k}}$ converges in the C^1 topology to a function $\tilde{\psi}$, then $\tilde{\psi} = \psi$. Thus the formula (6.1) determines uniquely a local strongly stable manifold through z and the formula $W(z) = \cup_{n\geq 0} h^{-t}(V(h^t(z))$ defines the global strongly stable manifold through z. These manifolds form a continuous strongly stable foliation with smooth leaves for h^t. In a similar fashion we can obtain strongly unstable local manifolds and construct a strongly unstable foliation with smooth leaves for h^t. These two foliations are transverse at every point $z \in \mathcal{U}$.

We will now show that the Lyapunov exponent $\lambda^s_h(z)$ in the direction $E^s_h(z)$ is negative at almost every point $z \in \mathcal{U}$. Indeed, let $Z \subset \mathcal{U}$ be the set of points at which $\lambda^s_h(z) = 0$. If $m(Z) > 0$ then

$$\begin{aligned} 0 &= \int_Z \lambda^s_h(z)\, dm = \int_Z \lim_{n\to\infty} \frac{1}{n} \log \prod_{i=0}^{n-1} \lambda_h(h^i(z)) dm(z) \\ &= \lim_{n\to\infty} \frac{1}{n} \int_Z \sum_{i=0}^{n-1} \log \lambda_h(h^i(z))\, dm(z) \\ &= \int_Z \log \lambda_h(z)\, dm(z) < 0 \end{aligned}$$

(recall that $\lambda_h(z)$ is the contraction coefficient along $E^s_h(z)$). This contradiction proves our claim. Similarly, one can prove that the Lyapunov exponent $\lambda^u_h(z)$ in the direction $E^u_h(z)$ is positive at almost every point $z \in \mathcal{U}$.

Since h^t is nonuniformly partially hyperbolic on $\mathcal{U}$, by Theorem 8.6.1 in [2], we obtain that its strongly stable and unstable foliations are absolutely continuous.

Next we will show that the flow h^t has the accessibility property on $\mathcal{U}$ via its invariant foliations W^s_h and W^u_h. Indeed, by Proposition 5.1 (5), for any $n > k$, $\omega = s, u, c$ and any $z \in \mathcal{U}_k \subset \mathcal{U}$,

$$\angle(E^\omega_{h_n}(z), E^\omega_{h_k}(z)) \le \delta_k(1 - 1/2^{n-k}) < \delta_k.$$

Taking the limit as $n \to \infty$, we obtain that $\angle(E^\omega_h(z), E^\omega_{h_k}(z)) \le \delta_k$ on $\mathcal{U}_k$. Hence, by Proposition 5.1(6), the flow h^t has the accessibility property on each $\widetilde{\mathcal{U}}_k$. Since k is arbitrary, we obtain that the flow h^t has the accessibility property on $\mathcal{U}$.

To show that the flow h^t has positive central Lyapunov exponent, we first recall that the average Lyapunov exponents of the flow g^t are arranged as in (4.2). Set $c = L_3(g^t) - L_2(g^t) > 0$. By the upper semicontinuity of $L_i(\cdot)$, we choose the number $\delta_h > 0$ in Proposition 5.1 so small that $L_2(h^t) < L_2(g^t) + c/2$. On the other hand, it follows from Proposition 5.1(4) that

$$L_4(h^t_n) = \int_{\mathcal{M}} \det(dh^t_n | E^{uab\tau}_{h_n}(z))dm = \int_{\mathcal{M}} \det(dg^t | E^{uab\tau}_g(z))dm = L_4(g^t).$$

Taking the limit as $n \to \infty$ we obtain $L_4(h^t) = L_4(g^t) = L_3(g^t)$. Therefore,

$$L_4(h^t) - L_2(h^t) = \int_{\mathcal{M}} (\lambda_3(z, h^t) + \lambda_4(z, h^t))dm(z) \ge c/2 > 0,$$

then there is a subset $\mathcal{A} \subset \mathcal{U}$ such that $\lambda_3(z, h^t) + \lambda_4(z, h^t) > 0$ for all $z \in \mathcal{A}$, and thus $\lambda_2(z, h^t) \ge \lambda_3(z, h^t) \ge \frac{1}{2}[\lambda_3(z, h^t) + \lambda_4(z, h^t)] > 0$ for all $z \in \mathcal{A}$. Since the center subspace $E^c_h(z)$ is 3-dimensional and the flow direction $\mathrm{Span}\{\mathcal{X}_h\}$ corresponds to zero exponent, we conclude that $\lambda_2(z, h^t)$ and $\lambda_3(z, h^t)$ correspond to vectors in $E^c_h(z) \backslash \mathrm{Span}\{\mathcal{X}_h\}$. Thus the flow h^t has positive central Lyapunov exponents.

By Theorem 2.4, we obtain that h^t has positive central exponents at almost every point in $\mathcal{U}$, and $h^t|\mathcal{U}$ is an ergodic flow.

Appendix A. The differential and metric structures of the suspension manifold

We specify the differential and metric structure of the suspension manifold $\mathcal{N}$ and the 5-dimensional manifold $\mathcal{M}$ in Section 3. Associated to the Anosov automorphism A of $X = \mathbb{T}^2$, one can find smooth local charts (U_x, ϕ_x) around each $x \in X$ such that

$$\phi_x : U_x \to (-u_0(x), u_0(x)) \times (-s_0(x), s_0(x))$$

satisfies $\phi_x(x) = (0, 0)$ and

$$\phi_{Ax} \circ A \circ \phi_x^{-1}(u, s) = (\eta u, \eta^{-1} s),$$

where $u_0(x), s_0(x) > 0$ are sizes of charts depending on x, and $\eta > 1$ is the expanding rate along the unstable direction. In fact, $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial s}$ are the unstable and stable directions of A respectively.

Recall that the suspension manifold $\mathcal{N}$ is the quotient space $X \times \mathbb{R}/\sim$ with the equivalence relation $(x, \tau + 1) \sim (Ax, \tau)$. Let $\pi : X \times \mathbb{R} \to \mathcal{N}$ be the natural projection. Following [12] there is a natural differential structure on $\mathcal{N}$ with atlas $(U^1_{(x,\tau)}, \phi^1_{(x,\tau)})$ for $\tau \in (-1/4, 3/4)$ and $(U^2_{(x,\tau)}, \phi^2_{(x,\tau)})$ for $\tau \in (1/4, 5/4)$, where

$$U^1_{(x,\tau)} = \pi(U_x \times (-1/4, 3/4)), \quad \phi^1_{(x,\tau)}(\pi(\phi_x^{-1}(u, s), \tau)) = (u, s, \tau);$$
$$U^2_{(x,\tau)} = \pi(U_x \times (\ \ 1/4, 5/4)), \quad \phi^2_{(x,\tau)}(\pi(\phi_x^{-1}(u, s), \tau)) = (u, s, \tau).$$

It is easy to verify that

$$\phi^i_{(x',\tau')} \circ \phi^i_{(x,\tau)}(\pi(\phi_x^{-1}(u, s), \tau) = (\phi_{x'} \circ \phi_x^{-1}(u, s), \tau), \quad i = 1, 2;$$
$$\phi^1_{(x',\tau')} \circ \phi^2_{(x,\tau)}(\pi(\phi_x^{-1}(u, s), \tau) = (\phi_{Ax'} \circ A \circ \phi_x^{-1}(u, s), \tau - 1).$$

In particular,

$$\phi^1_{(x,\tau')} \circ \phi^2_{(x,\tau)}(\pi(\phi_x^{-1}(u, s), \tau) = (\eta u, \eta^{-1} s, \tau - 1). \tag{A.1}$$

There are three subbundles E^u, E^s and E^τ on $\mathcal{N}$ generated by independent vector fields $d\pi(\frac{\partial}{\partial u})$, $d\pi(\frac{\partial}{\partial s})$ and $d\pi(\frac{\partial}{\partial \tau})$ respectively. By [3], we can choose the Riemannian metric on $\mathcal{N}$ which has the local representation $\eta^{2\tau} du^2 + \eta^{-2\tau} ds^2 + d\tau^2$. Under this metric, the suspension flow $S^t : \mathcal{N} \to \mathcal{N}$ satisfies

$$\begin{aligned} \|dS^t v\| &= \eta^t \|v\|, & v \in E^u, \\ \|dS^t v\| &= \eta^{-t} \|v\|, & v \in E^s, \\ \|dS^t v\| &= \|v\|, & v \in E^\tau. \end{aligned}$$

For the 2-torus $Y = \mathbb{T}^2$, we choose the local coordinate (a, b) centered at each $y \in Y$. Given $z = (x, \tau, y) \in \mathcal{M} = \mathcal{N} \times Y$, we can hence choose a local coordinate system (u, s, τ, a, b), endowed with the product Riemannian metric $\eta^{2\tau}du^2 + \eta^{-2\tau}ds^2 + d\tau^2 + da^2 + db^2$. In particular, the metric on the cross-section $X \times \{0\} \times Y$ is given by the flat metric

$$du^2 + ds^2 + da^2 + db^2. \tag{A.2}$$

Appendix B. Partial Hyperbolicity of the Limit Flow

Let $\mathcal{M}$ be a compact smooth Riemannian manifold and $\mathcal{S} \subset \mathcal{M}$ an open subset. Let also H^t be the flow on $\mathcal{M}$ that is pointwise partially hyperbolic on $\mathcal{S}$. Further, let $\mathcal{U}_n \subset \mathcal{S}$, $n \geq 1$ be a sequence of open subsets such that:

(1) $\mathcal{U}_n \subset \overline{\mathcal{U}_n} \subset \mathcal{U}_{n+1}$ and $\bigcup \mathcal{U}_n = \mathcal{S}$;
(2) each $\mathcal{U}_n$ is H^t-invariant;
(3) $H^t|\overline{\mathcal{U}_n}$ is uniformly partially hyperbolic.

The goal of this appendix is to prove the following statement.

Theorem B.1. *There exists a sequence of positive numbers ε_n such that if smooth vector fields $\mathcal{X}_n$ on $\mathcal{M}$ satisfy*

$$\mathcal{X}_0 = \mathcal{X}_H, \quad \mathcal{X}_n = \mathcal{X}_{n-1} \text{ on } \mathcal{M}\backslash\overline{\mathcal{U}_n}$$

and

$$\|\mathcal{X}_n - \mathcal{X}_{n-1}\|_{C^1} \leq \varepsilon_n,$$

then for every $n \geq 1$ the corresponding flow h_n^t is uniformly partially hyperbolic on the invariant set $\overline{\mathcal{U}_n}$ and hence pointwise partially hyperbolic on $\mathcal{S}$. Moreover, the limit vector field $\mathcal{X} = \lim_{n\to\infty} \mathcal{X}_n$ is of class C^1 and generates a pointwise partially hyperbolic flow h^t on $\mathcal{S}$.

We need the following technical statements.

Lemma B.2. *Given a sequence of positive numbers $\{a_n\}_{n\geq 1}$ satisfying $\sum_{n=1}^{\infty} a_n \leq \frac{1}{4}$, we have*

$$\prod_{n=1}^{\infty}(1 + a_n) \leq 1 + 2\sum_{n=1}^{\infty} a_n, \quad \prod_{n=1}^{\infty}(1 - a_n) \geq 1 - 2\sum_{n=1}^{\infty} a_n.$$

Lemma B.3 (Gronwall's inequality)**.** *Let $\eta(t)$ be a nonnegative C^1 function on $[0, T]$ satisfying*

$$\eta'(t) \leq \phi(t)\eta(t) + \psi(t),$$

where $\phi(t)$ and $\psi(t)$ are nonnegative integrable functions, then for all $0 \leq t \leq T$,

$$\eta(t) \leq e^{\int_0^t \phi(s)ds}[\eta(0) + \int_0^t \psi(s)ds].$$

Lemma B.4. *Set* $K := 2\|\mathcal{X}_H\|_{C^1}$. *If* $\varepsilon_n < \frac{K}{2^{n+1}}$, *then* $\|\mathcal{X}_n\|_{C^1} \le K$ *for all* $n \ge 0$. *Moreover, given a flow* F^t *with* $\|\mathcal{X}_F\|_{C^1} \le K$, *we have for any* $x \in \mathcal{M}$ *and* $t \in \mathbb{R}^+$ *that,*

$$e^{-tK} \le m(d_xF^t) \le \|d_xF^t\| \le e^{tK}.$$

In particular,

$$e^{-K} \le \min_{0\le t\le 1} m(d_xF^t) \le \max_{0\le t\le 1} \|d_xF^t\| \le e^K.$$

Proof of the lemma. Since $\|\mathcal{X}_H\|_{C^1} = K/2$, we find that

$$\|\mathcal{X}_n - \mathcal{X}_H\|_{C^1} \le \sum_{k=1}^n \|\mathcal{X}_k - \mathcal{X}_{k-1}\|_{C^1} \le \sum_{k=1}^n \varepsilon_k < K/2.$$

This implies that $\|\mathcal{X}_n\|_{C^1} \le K$.

Let F^t be a flow and $\mathcal{X}_F$ the corresponding vector field. Consider the variational differential equation

$$\frac{d}{dt} d_xF^t = D\mathcal{X}_F(F^t(x))d_xF^t,$$

for any $x \in \mathcal{M}$ and $t \in \mathbb{R}^+$. Then

$$\frac{d}{dt}\|d_xF^t\| \le \|\frac{d}{dt}d_xF^t\| \le \|D\mathcal{X}_F\|\|d_xF^t\| \le K\|d_xF^t\|.$$

Since $\|d_xF^0\| = 1$, by Lemma B.3, we obtain that $\|d_xF^t\| \le e^{tK}$. Noting that $m(d_xF^t) = \|d_xF^{-t}\|^{-1}$ and the flow F^{-t} corresponds to the vector field $-\mathcal{X}_F$, we get that $m(d_xF^t) \ge e^{-tK}$. □

Let F^t and G^t be flows on $\mathcal{M}$ and $\mathcal{X}_F$ and $\mathcal{X}_G$ the corresponding vector fields. Assume that $\|\mathcal{X}_F\|_{C^1}, \|\mathcal{X}_G\|_{C^1} \le K$, where K is given in Lemma B.4.

Lemma B.5. *Set*

$$M := e^{3K}\|\mathcal{X}_F\|_{C^2} + e^{2K}, \quad \varepsilon_{F,G} := \|\mathcal{X}_G - \mathcal{X}_F\|_{C^1}.$$

Then for $t \in [0,1]$,

$$\rho_{C^1}(G^t, F^t) \le 2tM\varepsilon_{F,G}, \tag{B.1}$$

where $\rho_{C^1}(G^t, F^t) = \max_{x\in\mathcal{M}}(\mathrm{dist}(G^t(x), F^t(x)) + \|d_xF^t - d_xG^t\|)$ *is the distance between the flows in the* C^1 *topology.*

Proof of the lemma. Consider the family of flows $F^t(\tau)$ generated by the family of vector fields $(1-\tau)\mathcal{X}_F + \tau\mathcal{X}_G$ with $\tau \in [0,1]$. Given $x \in \mathcal{M}$ and $t \in [0,1]$, the curve $c_t = c_t^x : \tau \mapsto F^t(\tau)(x)$ is of length

$$L(c_t) = \int_0^1 \left\|\frac{\partial c_t}{\partial \tau}\right\| d\tau = \int_0^1 \left\|\frac{\partial}{\partial \tau}F^t(\tau)(x)\right\| d\tau,$$

and hence,

$$\begin{aligned}\frac{d}{dt}L(c_t) &\leq \int_0^1 \left\| \frac{\partial}{\partial \tau}\frac{d}{dt}F^t(\tau)(x)\right\| d\tau \\ &\leq \|\mathcal{X}_G - \mathcal{X}_F\| + [(1-\tau)\|D\mathcal{X}_F\| + \tau\|D\mathcal{X}_G\|]\int_0^1 \left\|\frac{\partial c_t}{\partial \tau}\right\| d\tau \\ &\leq \varepsilon_{F,G} + KL(c_t).\end{aligned}$$

Recall that $L(c_0) = 0$, $c_t(0) = F^t x$ and $c_t(1) = G^t x$. By Lemma B.3, we obtain

$$\operatorname{dist}(F^t x, G^t x) \leq L(c_t) \leq te^{tK}\varepsilon_{F,G} \leq tM\varepsilon_{F,G}. \tag{B.2}$$

On the other hand,

$$\begin{aligned}\frac{d}{dt}\|d_xF^t - d_xG^t\| &\leq \left\|D\mathcal{X}_F(F^tx)d_xF^t - D\mathcal{X}_G(G^tx)d_xG^t\right\| \\ &\leq \left\|D\mathcal{X}_F(F^tx)d_xF^t - D\mathcal{X}_F(F^tx)d_xG^t\right\| \\ &+ \left\|D\mathcal{X}_F(F^tx)d_xG^t - D\mathcal{X}_F(G^tx)d_xG^t\right\| \\ &+ \left\|D\mathcal{X}_F(G^tx)d_xG^t - D\mathcal{X}_G(G^tx)d_xG^t\right\| \\ &\leq \|D\mathcal{X}_F\|\|d_xF^t - d_xG^t\| \\ &+ \|D^2\mathcal{X}_F\|\operatorname{dist}(F^tx, G^tx)\|d_xG^t\| \\ &+ \|D\mathcal{X}_F - D\mathcal{X}_G\|\|d_xG^t\| \\ &\leq K\|d_xF^t - d_xG^t\| + Me^{-K}\varepsilon_{F,G},\end{aligned}$$

where in the last inequality we use Lemma B.4 and the inequalities (B.2). By Lemma B.3, we obtain

$$\|d_xF^t - d_xG^t\| \leq tMe^{(t-1)K}\varepsilon_{F,G} \leq tM\varepsilon_{F,G}. \tag{B.3}$$

Now (B.1) follows by combining (B.2) and (B.3). □

Given flows F^t and G^t and invariant distributions E_F and E_G on $\mathcal{S}$ respectively, let

(B.4)

$$\begin{aligned}\Delta_{F^t,G^t,E_F,E_G}(x) &= \max\left\{\left|\frac{\|d_xG^t|E_G(x)\|}{\|d_xF^t|E_F(x)\|} - 1\right|, \left|\frac{m(d_xG^t|E_G(x))}{m(d_xF^t|E_F(x))} - 1\right|\right\}, \\ \delta_{F^t,G^t} &= \|G^t - F^t\|_{C^1}, \quad \theta_{E_F,E_G}(x) = \angle(E_F(x), E_G(x)).\end{aligned}$$

Lemma B.6. *Assume that* $\|\mathcal{X}_F\|_{C^1} \leq K$, *then*

$$\Delta_{F^t,G^t,E_F,E_G}(x) \leq e^K[\delta_{F^t,G^t} + Ce^K\theta_{E_F,E_G}(x)]$$

for any $x \in \mathcal{S}$ *and* $t \in [0,1]$, *where* $C > 0$ *is a constant which depends only on the Riemannian metric of* $\mathcal{M}$.

Proof of the lemma. Similarly to the proof of Lemma B.4, we can show that if $\|\mathcal{X}_F\|_{C^1} \le K$ then for any $x \in \mathcal{S}$ and $t \in [0,1]$,

$$e^{-K} \le m(d_x F^t) \le \|d_x F^t\| \le e^K.$$

We have that

$$\begin{aligned}
|\|d_xG^t|E_G(x)\| - \|d_xF^t|E_F(x)\|| &\le \left|\|d_xG^t|E_G(x)\| - \|d_xF^t|E_G(x)\|\right| \\
&\quad + \left|\|d_xF^t|E_G(x)\| - \|d_xF^t|E_F(x)\|\right| \\
&\le \|d_xG^t - d_xF^t\| + \|d_xF^t\|\mathrm{dist}(E_G(x), E_F(x)) \\
&\le \|d_xG^t - d_xF^t\| + C\|d_xF^t\|\angle(E_G(x), E_F(x))
\end{aligned}$$

for some constant $C > 0$ depending only on the Riemannian metric of $\mathcal{M}$. Dividing both sides of the inequality by $\|d_xF^t|E_F(x)\|$ and noting that $\|d_xF^t|E_F(x)\| \ge m(d_xF^t)$, we obtain that

$$\begin{aligned}
\left|\frac{\|d_xG^t|E_G(x)\|}{\|d_xF^t|E_F(x)\|} - 1\right| &\le \frac{1}{m(d_xF^t)}[\|d_xG^t - d_xF^t\| \\
&\quad + C\|d_xF^t\|\angle(E_G(x), E_F(x))] \\
&\le e^K[\delta_{F^t,G^t} + Ce^K\theta_{E_F,E_G}(x)].
\end{aligned}$$

Similarly, one can show that $\left|\dfrac{m(d_xG^t|E_G(x))}{m(d_xF^t|E_F(x))} - 1\right|$ admits the same upper bound. □

Lemma B.7. *A flow F^t is uniformly partially hyperbolic on a compact invariant subset $\Lambda \subset \mathcal{S}$ if and only if the time-1 map $F^1|\Lambda$ is uniformly partially hyperbolic.*

Proof of the lemma. See [9]. □

Lemma B.8. *Suppose that F^t is uniformly partially hyperbolic on a compact invariant subset $\Lambda \subset \mathcal{S}$. Pick numbers $0 < \lambda < \widetilde{\lambda} \le 1 \le \widetilde{\mu} < \mu$ such that*

$$\begin{aligned}
\lambda \ge \lambda(F^1,\Lambda) = \sup_{x\in\Lambda}\|d_x^sF^1\|, &\qquad \widetilde{\lambda} \le \widetilde{\lambda}(F^1,\Lambda) = \inf_{x\in\Lambda} m(d_x^cF^1), \\
\widetilde{\mu} \ge \widetilde{\mu}(F^1,\Lambda) = \sup_{x\in\Lambda}\|d_x^cF^1\|, &\qquad \mu \le \mu(F^1,\Lambda) = \inf_{x\in\Lambda} m(d_x^uF^1),
\end{aligned}$$

where $d_x^\omega F^t = d_xF^t|E_F^\omega(x)$, $\omega = s,c,u$. Given $\Delta > 0$, there is $\varepsilon = \varepsilon(\Delta, \lambda, \widetilde{\lambda}, \widetilde{\mu}, \mu)$ such that if $\|\mathcal{X}_G - \mathcal{X}_F\|_{C^1} < \varepsilon$ and $\mathcal{X}_G = \mathcal{X}_F$ on $\mathcal{S}\backslash\Lambda$, then $G^t|\Lambda$ is also a uniformly partially hyperbolic flow and

(B.5)
$$\Delta^\omega_{F^t,G^t}(x) := \Delta_{F^t,G^t,E_F^\omega,E_G^\omega}(x) \le \Delta t, \quad \omega = s,c,u,\ x \in \Lambda,\ t \in [0.5,1].$$

In particular,

$$(B.6)\quad 1-\Delta \le \frac{\lambda(G^1,\Lambda)}{\lambda(F^1,\Lambda)},\ \frac{\widetilde{\lambda}(G^1,\Lambda)}{\widetilde{\lambda}(F^1,\Lambda)},\ \frac{\widetilde{\mu}(G^1,\Lambda)}{\widetilde{\mu}(F^1,\Lambda)},\ \frac{\mu(G^1,\Lambda)}{\mu(F^1,\Lambda)} \le 1+\Delta.$$

Proof of the lemma. Consider the time-1 map F^1. By [13], there is $\varepsilon < \Delta e^{-K}/4M$ depending on $\Delta, \lambda, \widetilde{\lambda}, \widetilde{\mu}, \mu$ such that if $\|\mathcal{X}_G - \mathcal{X}_F\|_{C^1} < \varepsilon$ and $\mathcal{X}_G = \mathcal{X}_F$ on $\mathcal{S}\backslash\Lambda$, then $G^1|\Lambda$ is uniformly partially hyperbolic on Λ with

$$(B.7)\qquad \sup_{x\in\Lambda} \angle(E^\omega_{G^1}(x), E^\omega_{F^1}(x)) < \frac{\Delta}{4Ce^{2K}}.$$

By Lemma B.7, the flow G^t is uniformly partially hyperbolic on Λ with the same invariant distributions as its time-1 map G^1. Moreover, it follows from Lemma B.5 and B.6 that

$$\Delta_{F^t,G^t,E^\omega_F,E^\omega_G}(x) \le \frac{\Delta t}{2} + \frac{\Delta}{4} \le \Delta t, \quad \omega = s,c,u,\ x\in\Lambda,\ t\in[0.5,1].$$

In particular,

$$\|d^s_x G^1\| \le \|d^s_x F^1\|(1+\Delta) \le \lambda(1+\Delta),$$

and hence $\dfrac{\lambda(G^1,\Lambda)}{\lambda(F^1,\Lambda)} \le 1+\Delta$. The other inequalities in (B.6) can be shown in a similar fashion. □

We will now specify how to choose the sequence of numbers ε_n in the theorem. First choose four sequences of numbers $0 < \lambda_n < \widetilde{\lambda}_n \le 1 \le \widetilde{\mu}_n < \mu_n$ such that

(1) $\lambda_n \ge \lambda(H^1,\overline{\mathcal{U}_n})$, $\widetilde{\lambda}_n \le \widetilde{\lambda}(H^1,\overline{\mathcal{U}_n})$, $\widetilde{\mu}_n \ge \widetilde{\mu}(H^1,\overline{\mathcal{U}_n})$, $\mu_n \le \mu(H^1,\overline{\mathcal{U}_n})$;

(2) $\lambda_n, \widetilde{\mu}_n$ are strictly increasing while $\widetilde{\lambda}_n, \mu_n$ are strictly decreasing.

For all $x\in\mathcal{S}$, let

$$\gamma(x) = \min\left\{\frac{\min\{1, m(d^c_x H^1)\}}{\|d^s_x H^1\|},\ \frac{m(d^u_x H^1)}{\max\{1, \|d^c_x H^1\|\}}\right\},$$

and choose a strictly decreasing sequence of numbers γ_n such that

$$(B.8)\qquad 0 < \gamma_n \le \inf_{x\in\overline{\mathcal{U}_n}} \frac{\gamma(x)-1}{8}.$$

Now choose a sequence of positive numbers Δ_n such that

$$(B.9)\quad \max\{\frac{\widetilde{\lambda}_{n+1}}{\widetilde{\lambda}_n},\ \frac{\mu_{n+1}}{\mu_n}\} \le 1-\Delta_n < 1+\Delta_n \le \min\{\frac{\lambda_{n+1}}{\lambda_n},\ \frac{\widetilde{\mu}_{n+1}}{\widetilde{\mu}_n}\};$$

$$\Delta_n < \frac{1}{2^{n+2}}, \quad \sum_{k=n}^{\infty} \Delta_k < \gamma_n. \tag{B.10}$$

Finally, choose

$$\varepsilon_n < \frac{1}{2} \min\{\frac{K}{2^{n+1}}, \varepsilon(\Delta_n, \lambda_n, \widetilde{\lambda}_n, \widetilde{\mu}_n, \mu_n)\},$$

where $\varepsilon(\Delta, \lambda, \widetilde{\lambda}, \widetilde{\mu}, \mu)$ is given by Lemma B.8.

Proof of Theorem B.1. First we will show that for every $n > 0$ the map h_n^t is uniformly partially hyperbolic on $\overline{\mathcal{U}_n}$. This is clearly true for h_0^t and we will use induction assuming that $h_k^t|\overline{\mathcal{U}_k}$ for $k = 1, \dots, n-1$ are uniformly partially hyperbolic. By Lemma B.6, we obtain that

$$1-\Delta_k \leq \frac{\lambda(h_k^1,\overline{\mathcal{U}_k})}{\lambda(h_{k-1}^1,\overline{\mathcal{U}_k})}, \frac{\widetilde{\lambda}(h_k^1,\overline{\mathcal{U}_k})}{\widetilde{\lambda}(h_{k-1}^1,\overline{\mathcal{U}_k})}, \frac{\widetilde{\mu}(h_k^1,\overline{\mathcal{U}_k})}{\widetilde{\mu}(h_{k-1}^1,\overline{\mathcal{U}_k})}, \frac{\mu(h_k^1,\overline{\mathcal{U}_k})}{\mu(h_{k-1}^1,\overline{\mathcal{U}_k})} \leq 1+\Delta_k.$$

Note that

$$\begin{aligned}\lambda(h_k^1,\overline{\mathcal{U}_{k+1}}) &\leq \max\{\lambda(H^1,\overline{\mathcal{U}_{k+1}}), \lambda(h_k^1,\overline{\mathcal{U}_k})\} \\ &\leq \max\{\lambda_{k+1}, \lambda(h_k^1,\overline{\mathcal{U}_k})\} \\ &\leq \max\{\lambda_{k+1}, \lambda(h_{k-1}^1,\overline{\mathcal{U}_k})(1+\Delta_k)\}.\end{aligned}$$

The fact that $\lambda(h_0^1,\overline{\mathcal{U}_1}) \leq \lambda_1$ and the choice of Δ_n in (B.9) guarantee that

$$\lambda_n' := \lambda(h_{n-1}^1,\overline{\mathcal{U}_n}) \leq \lambda_n.$$

Similarly, we have

$$\widetilde{\lambda}_n' := \widetilde{\lambda}(h_{n-1}^1,\overline{\mathcal{U}_n}) \geq \widetilde{\lambda}_n, \quad \widetilde{\mu}_n' := \widetilde{\mu}(h_{n-1}^1,\overline{\mathcal{U}_n}) \leq \widetilde{\mu}_n,$$

$$\mu_n' := \mu(h_{n-1}^1,\overline{\mathcal{U}_n}) \geq \mu_n.$$

It follows that

$$\varepsilon_n \leq \varepsilon(\Delta_n, \lambda_n, \widetilde{\lambda}_n, \widetilde{\mu}_n, \mu_n) \leq \varepsilon(\Delta_n, \lambda_n', \widetilde{\lambda}_n', \widetilde{\mu}_n', \mu_n').$$

Since $\|\mathcal{X}_n - \mathcal{X}_{n-1}\|_{C^1} \leq \varepsilon_n$, by Lemma B.8, we obtain that $h_n^t|\overline{\mathcal{U}_n}$ is uniformly partially hyperbolic.

Next we will show that $\mathcal{X} = \lim_{n\to\infty} \mathcal{X}_n$ exists and is smooth. In fact, $\{\mathcal{X}_n\}$ is a Cauchy sequence in the C^1 topology since for any $n, m \in \mathbb{N}$,

$$\|\mathcal{X}_{n+m} - \mathcal{X}_n\|_{C^1} \leq \sum_{l=1}^{m} \|\mathcal{X}_{n+l} - \mathcal{X}_{n+l-1}\|_{C^1} \leq \sum_{l=1}^{m} \varepsilon_{n+l} \leq \frac{K}{2^{n+1}}.$$

Hence $\mathcal{X} = \lim_{n\to\infty} \mathcal{X}_n$ exists and is C^1.

It remains to show that the flow h^t generated by $\mathcal{X}$ is pointwise partially hyperbolic on $\mathcal{S}$. First we construct invariant distributions for h^t. Given $x \in \mathcal{S}$, we have

$$\angle(E^\omega_{h_n}(x), E^\omega_{h_{n-1}}(x)) \le \frac{\Delta_n}{4Ce^{2K}} < \frac{1}{2^{n+4}Ce^{2K}}, \quad \omega = s, c, u.$$

Hence the sequence of subspaces $E^\omega_{h_n}(x)$ is Cauchy and it converges to

$$E^\omega_h(x) = \lim_{n\to\infty} E^\omega_{h_n}(x),$$

which is clearly dh^t-invariant for all $t \in \mathbb{R}^+$.

Now we would like to estimate $\Delta^\omega_{h^1,H^1}(x)$. Fix $x \in \mathcal{U}_n \backslash \overline{\mathcal{U}_{n-1}}$, we have

$$\Delta^\omega_{h^1_k,h^1_{k-1}}(x) \begin{cases} = 0, & k < n, \\ \le \Delta_k, & k \ge n. \end{cases}$$

Note that

$$\frac{\|d^\omega_x h^1_l\|}{\|d^\omega_x H^1\|} = \prod_{k=1}^{l} \frac{\|d^\omega_x h^1_k\|}{\|d^\omega_x h^1_{k-1}\|}, \qquad \frac{m(d^\omega_x h^1_l)}{m(d^\omega_x H^1)} = \prod_{k=1}^{l} \frac{m(d^\omega_x h^1_k)}{m(d^\omega_x h^1_{k-1})},$$

and $\sum \Delta_k < 1/4$, we obtain by Lemma B.2,

$$\Delta^\omega_{h^1_l,H^1}(x) \le \prod_{k=1}^{l}(1 + \Delta^\omega_{h^1_k,h^1_{k-1}}(x)) - 1 \le \prod_{k=n}^{\infty}(1+\Delta_k) - 1 \le 2\sum_{k=n}^{\infty}\Delta_k.$$

Letting $l \to \infty$, we have

$$\Delta^\omega_{h^1,H^1}(x) \le 2\sum_{k=n}^{\infty}\Delta_k, \ \omega = s, c, u, \ x \in \mathcal{U}_n\backslash\overline{\mathcal{U}_{n-1}}.$$

Therefore,

$$\begin{aligned}\frac{\|d^s_x h^1\|}{\min\{1, m(d^c_x h^1)\}} &\le \frac{1 + 2\sum_{k=n}^\infty \Delta_k}{1 - 2\sum_{k=n}^\infty \Delta_k} \frac{\|d^s_x H^1\|}{\min\{1, m(d^c_x H^1)\}} \\ &< (1 + 8\gamma_n)\frac{\|d^s_x H^1\|}{\min\{1, m(d^c_x H^1)\}} \\ &\le \gamma(x)\frac{\|d^s_x H^1\|}{\min\{1, m(d^c_x H^1)\}} < 1.\end{aligned}$$

Similarly, one can show that $m(d^u_x h^1) > \max\{1, \|d^c_x h^1\|\}$. It follows that h^1 is pointwise partially hyperbolic on $\mathcal{S}$, and so is the flow h^t by definition. □

References

[1] K. Burns, D. Dolgopyat and Ya. Pesin, *Partial hyperbolicity, Lyapunov exponents and stable ergodicity*, J. Stat. Phys., **108** no. 5-6 (2002) 927-942.

[2] L. Barreira and Ya. Pesin, *Nonuniform Hyperbolicity: Dynamics of Systems with Nonzero Lyapunov Exponents*, Encyclopedia of Mathematics and Its Applications, **115**,Cambridge Univ. Press, 2007.

[3] A. Bolsinov and I. Taimanov, *Integrable geodesic flows with positive topological entropy*, Invent. Math., **140:3** (2000) 639 - 650.

[4] K. Burns and A. Wilkinson, *On the ergodicity of partially hyperbolic systems*, Annals of Math **171** (2010) 451–489.

[5] J. Chen, H. Hu and Ya. Pesin *The essential coexistence phenomenon in dynamics* Dynamical Systems: An International Journal, 2013 (to apear).

[6] D. Dolgopyat, H. Hu and Ya. Pesin, *An example of a smooth hyperbolic measure with countably many ergodic components*, in Smooth Ergodic Theory and Its Applications (Seattle, 1999), Proc. Sympos. Pure Math., **69**, Amer. Math. Soc. Providence, RI, 2001.

[7] D. Dolgopyat and Ya. Pesin, *Every compact manifold carries a completely hyperbolic diffeomorphism*, Ergodic Theory Dyn. Systems, **22** (2002) 409–435.

[8] V. Donnay, *Geodesic flow on the two-sphere. I. Positive measure entropy*, Ergodic Theory Dyn. Systems **8**, no. 4 (1988), 531553.

[9] M. Hirsch, C. Pugh and M. Shub, *Invariant manifolds*, Lecture Notes in Mathematics, **583**, Springer-Verlag, Berlin, 1977, p. 154.

[10] H. Hu, Ya. Pesin and A. Talitskaya, *A volume preserving diffeomorphism with essential coexistence of zero and nonzero Lyapunov exponents*, Comm. Math. Phys, 2012 (to appear).

[11] I. Kornfeld, Ya. Sinai and S. Fomin, *Ergodic theory*, Springer-Verlag, New York, 1982.

[12] J. Palis and W. de Melo, *Geometric theory of dynamical systems*, Springer-Verlag, Berlin, New York, 1982.

[13] Ya. Pesin, *Lectures on partial hyperbolicity and stable ergodicity*, Zürich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich (2004).

[14] C. Pugh and M. Shub, *Stably ergodic dynamical systems and partial hyperbolicity*, J. of Complexity **13** (1997), 125–179.

[15] M. Shub and A. Wilkinson, *Pathological foliations and removable zero exponents*, Invent. Math. **139** (2000), no. 3, 495–508.

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