Some Ergodic Properties of Commuting Diffeomorphisms

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Abstract. For a smooth $\mathbb{Z}^2-$action on a C^∞ compact Riemannian manifold M, we discuss its ergodic properties which include the decomposition of the tangent space of M into subspaces related to Lyapunov exponents, the existence of Lyapunov charts, and the subaddtivity of entropies.

§0 Introduction

In this paper we discuss some ergodic properties of commuting diffeomorphisms on a C^∞ compact Riemannian manifold concerning Lyapunov exponents and entropies. Let M be a compact C^∞ Riemannian manifold without boundary, $f, g \in \mathrm{Diff}^2(M)$ with $fg = gf$, where fg denote the composition of f and g. In fact f and g generate a smooth $\mathbb{Z}^2-$action on M. We will give a decomposition of the tangent space of M into subspaces related to the Lyapunov exponents of both actions f and g, and construct a family of Lyapunov charts. We will show that for almost every x in M, if f and g have same unstable subspace, then they have same unstable manifold at x. We will investigate the subaddtivity of entropies of commuting diffeomorphisms, i.e. the entropy of the composition fg is less than or equal to the sum of the entropies of f and g. In the circumstances for measure-theoretic entropies the subaddtivity always holds whenever the measure is invariant under the actions f and g, and becomes additive if the unstable subspace of one map does not intersect with the stable subspace of another map at almost every point, but for topological entropies additional condition is needed to obtain the subaddtivity.

We denote by $\mathcal{M}(M, f)$ the set of $f-$invariant Borel probability measures on M. It is known by many authors (for example, see Proposition 1.2 for the proof,) that f and g have common invariant measures, i.e. $\mathcal{M}(M, f) \cap \mathcal{M}(M, g) \neq \emptyset$. We let $\mathcal{M}(M, f, g) = \mathcal{M}(M, f) \cap \mathcal{M}(M, g)$.

Throughout the paper, we always assume that M is a compact C^∞ Riemannian manifold without boundary, f and g are C^2 diffeomorphism on M with $fg = gf$, μ is an $f-$ and $g-$invariant Borel probability measure on M, i.e. $\mu \in \mathcal{M}(M, f, g)$.

Let T_xM be the tangent space of M at $x \in M$. The diffeomorphism f induces a map $Df_x : T_xM \to T_{fx}M$. It is well known (see [O]) that there exists a measurable set Γ_f with $\nu\Gamma_f = 1$, $\forall\ \nu \in M(M, f)$, such that for all $x \in \Gamma_f$, $u \in T_xM$, the limit

$$\chi(x, u, f) = \lim_{n\to\infty} \frac{1}{n} \log \|Df_x^n u\|$$

exists and is called Lyapunov exponent of u at x. Let $\lambda_1(x, f) > \cdots > \lambda_{r(x,f)}(x, f)$ denote all Lyapunov exponents of f at x with multiplicities $m_1(x, f), \cdots, m_{r(x,f)}(x, f)$ respectively, and $T_xM = \bigoplus_{i=1}^{r(x,f)} E_i(x, f)$ be the corresponding decomposition of tangent space at $x \in M$.

Similarly, for diffeomorphism g we have Lyapunov exponents $\lambda_1(x,g) > \cdots > \lambda_{r(x,g)}(x,g)$ with multiplicities $m_1(x,g), \cdots, m_{r(x,g)}(x,g)$ respectively, and the corresponding decomposition $T_xM = \bigoplus_{j=1}^{r(x,g)} E_j(x,g)$.

Suppose f and g are commuting diffeomorphisms. The spectrum $\{\lambda_i(x,f), m_i(x,f)\}$ of f is $f-$invariant. We will show that it is also $g-$invariant, i.e. $\forall u \in E_i(x,f)$,

$$\chi(gx, Dg_x u, f) = \chi(x,u,f) = \lambda_i(x,f).$$

and therefore $Dg_x E_i(x,f) = E_i(gx,f)$. Thus we can redecompose each $E_i(x,f)$ according to diffeomorphism g and get the following.

Theorem A. *let M be a C^∞ compact Riemannian manifold without boundary, $f,g \in \mathrm{Diff}^2(M)$ with $fg=gf$. Then there exists a measurable set Γ with $f^s g^t \Gamma = \Gamma$, $\forall\ s,t \in \mathbb{Z}$ and $\mu\Gamma = 1$, $\forall\ \mu \in M(M,f,g)$, satisfying that for all $x \in \Gamma$, there is a decomposition of the tangent space into*

$$T_xM = \bigoplus_{i=1}^{r(x,f)} \bigoplus_{j=1}^{r(x,g)} E_{ij}(x),$$

such that $\forall\ s,t \in \mathbb{Z}$, if $E_{ij}(x) \neq \{0\}$, then $\forall\ 0 \neq u \in E_{ij}(x)$,

$$\lim_{n\to\infty} \frac{1}{n} \log \|D(f^s g^t)^n_x u\| = s\lambda_i(x,f) + t\lambda_j(x,g),$$

and if $(i_1,j_1) \neq (i_2,j_2)$, $E_{i_1j_1}(x) \neq \{0\}$, $E_{i_2j_2}(x) \neq \{0\}$, then

$$\lim_{n\to\infty} \frac{1}{n} \log \left|\sin(E_{i_1j_1}((f^s g^t)^n x), E_{i_2j_2}((f^s g^t)^n x))\right| = 0.$$

Moreover, $\forall\ s,t \in \mathbb{Z}$,
$D(f^s g^t)_x(E_{ij}(x)) = E_{ij}(f^s g^t x)$,
$\lambda_i(f^s g^t x, f) = \lambda_i(x,f)$, $\quad \lambda_j(f^s g^t x, g) = \lambda_j(x,g)$.

Probably, this result is known. However, because of its importance for our discussion, we state it here and give the proof in §2 for completeness. In particular, if we take $s=1$ and $t=0$, then we have

$$\lim_{n\to\infty} \frac{1}{n} \log \|Df^n_x u\| = \lambda_i(x,f), \quad \forall\ 0 \neq u \in E_{ij}(x),$$

$$\lim_{n\to\infty} \frac{1}{n} \log |\sin(E_{i_1j_1}(f^n x), E_{i_2j_2}(f^n x))| = 0.$$

and $Df_x E_{ij}(x) = E_{ij}(fx)$, $\lambda_i(fx,f) = \lambda_i(x,f)$, and $\lambda_j(fx,g) = \lambda_j(x,g)$. Symmetrically, we have similar results concerning diffeomorphism g, if we take $s=0$ and $t=1$. The explicit statement is given in Proposition 2.8.

By the definition of Lyapunov exponents, given $\gamma > 0$, $\forall\ n,k \in \mathbb{Z}$, $\|Df^n_x u\| e^{-n\lambda_i(x,f)}$ and $\|Dg^k_x u\| e^{-k\lambda_j(x,g)}$, $u \in E_{ij}(x)$, are dominated by $C(x)^{\pm 1} e^{\pm n\gamma} \|u\|$ and $C(x)^{\pm 1} e^{\pm k\gamma} \|u\|$, respectively. We will show in §3 that $C(x)$ can be chosen such that $C(f^{\pm 1}x), C(g^{\pm 1}x) \leq C(x)e^{\gamma}$. This is a generalization of Pesin's theory ([P]) to commuting diffeomorphisms.

Based on the facts we can construct a family of Lyapunov charts, on which the maps $\tilde{f}$ and $\tilde{g}$, induced by f and g respectively, act approximately to the linear maps with eigenvalues $e^{\lambda_i(x,f)}$ and $e^{\lambda_j(x,g)}$ respectively. The result is stated in Proposition 4.1.

Let $E^s(x,f) = \bigoplus_{\lambda_i(x,f)<0} E_i(x,f)$, $E^u(x,f) = \bigoplus_{\lambda_i(x,f)>0} E_i(x,f)$, and if $\lambda_i(x,f) = 0$ for some i then let $E^c(x,f) = E_i(x,f)$. Also, let $E^{sc}(x,f) = E^s(x,f) \oplus E^c(x,f)$, $E^{uc}(x,f) = E^u(x,f) \oplus E^c(x,f)$. The unstable manifold for diffeomorphism f, say, is defined by

$$w^u(x,f) = \{y \in M : \limsup_{n\to\infty} \frac{1}{n} \log d(f^{-n}x, f^{-n}y) < 0\},$$

which is $f-$invariant. It is easy to see by the definition that it is also $g-$ invariant. From Theorem A we know that $E^u(x,f) = E^u(x,g)$ or $E^{sc}(x,f) = E^{sc}(x,g)$ is equivalent to $E^u(x,f) \cap E^{sc}(x,g) = \{0\} = E^u(x,g) \cap E^{sc}(x,f)$. If it holds at some $x \in \Gamma$, we will prove in §5 that the unstable manifolds $w^u(x,f)$ and $w^u(x,g)$ coincide at x.

The next topic in this paper is concerning the relationship among the entropies of f, g and fg. We will prove that if f and g are C^2 diffeomorphisms on a smooth compact manifold preserving a Borel probability measure μ, then $h_\mu(fg) \leq h_\mu(f) + h_\mu(g)$. It is not true for general measure preserving transformations on a probability space. There are some unpublished counterexamples, for instance, due to D. S. Ornstein and B. Weiss, and due to J.-P. Thouvenot, of two commuting measure preserving automorphisms S, T of probability space $(X, \mathcal{B}, \nu)$ with $h_\nu(S) = h_\nu(T) = 0$, but such that $h_\nu(ST) > 0$. However, in the smooth dynamical systems the subadditivity of measure-theoretic entropies holds.

From Ledrappier and Young's formula relating entropies, exponents and dimensions, we know that the entropy of a diffeomorphism on a smooth manifold is determined by the behavior of the map on its unstable manifold. If two commuting diffeomorphisms have same family of unstable manifolds, i.e. $w^u(x,f) = w^u(x,g)$, $\mu-$a.e., then we can construct an increasing partition subordinate to the unstable manifolds (see §6), and use it to get equality in above subadditivity formula. We combine the two results in the following theorem which we will prove in §7 and §9.

Theorem B. *Let M be a C^∞ compact Riemannian manifold without boundary, $f, g \in$ $\mathrm{Diff}^2(M)$, and $fg = gf$. Then*

$$h_\mu(fg) \leq h_\mu(f) + h_\mu(g), \quad \forall \mu \in \mathcal{M}(M,f,g),$$

where $h_\mu(\cdot)$ denotes the measure-theoretic entropy. Moreover, if $E^u(x,f) \cap E^s(x,g) = \{0\}$ and $E^s(x,f) \cap E^u(x,g) = \{0\}$, $\mu - a.e$, then the equality holds.

For topological entropies the the answer to the question whether $h(fg) \leq h(f) + h(g)$ is also negative in general. L. Wayne Goodwyn has a counterexample for the case, i.e. there exists a compact metric space X and two homeomorphisms S and T with $ST = TS$ such that $h(S) = h(T) = 0$ and $h(ST) > 0$(see [G]). Since topological entropy is the supremum of measure-theoretic entropy, from Theorem B we can prove the formula if some additional hypotheses are given on diffeomorphism fg.

Theorem C. *Let M,f,g is same as in Theorem B. If for fg, one of the following conditions holds:*

i) $fg \in \mathrm{Diff}^{\infty}(M)$, *or*

ii) fg *is expensive, or*

iii) fg *has finite number of ergodic measures with maximal entropy,*

then

$$h(fg) \leq h(f) + h(g).$$

This theorem will be proved in §10. In the section we will also present a counterexample to show that if f and g are homeomorphisms on a compact smooth manifold, then the result of Theorem C fails. We don't know whether the result still holds if the additional hypotheses on fg are removed.

§1. Ergodicity

In this section we will give the definition of ergodicity and discuss the properties for two commuting continuous maps on a compact metric space.

For a map T from a set X to itself, we denote by $\mathrm{Fix}(T)$ the set of fixed points of T.

Proposition 1.1. *If T and S are commuting maps on a set X, then $S(\mathrm{Fix}(T)) \subset \mathrm{Fix}(T)$.*

Proof. Take $x \in \mathrm{Fix}(T)$. Since $T(Sx) = S(Tx) = Sx$, we have $Sx \in \mathrm{Fix}(T)$. ∎

Let $\mathcal{M}(X)$ be the set of Borel probability measures on a compact metric space X, and T be a map on X. T induces a map T^* on $\mathcal{M}(X)$ by putting $T^*\mu = \mu \circ T^{-1}, \forall \mu \in \mathcal{M}(X)$. Thus $\mathrm{Fix}(T^*) = \mathcal{M}(X, T)$.

Proposition 1.2. *If T and S are commuting continuous maps on a compact metric space X, then $\mathcal{M}(X,T) \cap \mathcal{M}(X,S) \neq \emptyset$.*

Proof. By Proposition 1.1, $S^*(\mathcal{M}(X,T)) \subset \mathcal{M}(X,T)$. Since $\mathcal{M}(X,T)$ is a nonempty compact convex set in weak * topology and S^* is continuous, we know that S^* has a fixed point in $\mathcal{M}(X,T)$. ∎

We write $\mathcal{M}(X,T,S) = \mathcal{M}(X,T) \cap \mathcal{M}(X,S)$. Since both $\mathcal{M}(X,T)$ and $\mathcal{M}(X,S)$ are convex sets, $\mathcal{M}(X,T,S)$ is convex.

Definition. *Suppose T and S are continuous maps on a compact metric space X with $TS = ST$. A measure $\mu \in \mathcal{M}(X,T,S)$ is said to be $(T,S)-$ergodic if for any measurable set B with $\mu(T^{-1}B \triangle B) = 0 = \mu(S^{-1}B \triangle B)$, $\mu(B) = 0$ or $\mu(B) = 1$.*

$(T,S)-$ergodicity shares some properties with those of single transformation. For example, we give the following propositions whose proof is parallel to the case of one transformation (See [W], Chapter 1 and 6).

Proposition 1.3. *μ is $(T,S)-$ergodic iff any measurable function ϕ on X with $\phi(Tx) = \phi(x) = \phi(Sx)$, $\mu - a.e.$ is constant $\mu - a.e.$*

Proof. "$\Rightarrow$" is based on the fact that for such function ϕ, the set $\{x : \phi(x) > C\}$, $C \in \mathbb{R}$, is invariant under the actions T and S. "$\Leftarrow$" holds because the characteristic function

χ_B, where B is a set with $\mu(T^{-1}B\triangle B) = 0 = \mu(S^{-1}B\triangle B)$, satisfies $\chi_B(Tx) = \chi_B(x) = \chi_B(Sx)$, $\mu - a.e.$ and equals to 1 or 0 almost everywhere. ∎

Proposition 1.4. *μ is $(T,S)-$ergodic iff μ is an extreme point of $\mathcal{M}(X,T,S)$.*

Proof. Suppose μ is not $(T,S)-$ergodic. We can find a measurable set E with $TE = E = SE$ and $0 < \mu(E) < 1$. Let

$$\mu_1(\cdot) = \frac{\mu(\cdot \cap E)}{\mu(E)}, \qquad \mu_2(\cdot) = \frac{\mu(\cdot \cap X\backslash E)}{\mu(X\backslash E)}.$$

Then $\mu = p\mu_1 + (1-p)\mu_2$, where $p = \mu_1(E)$. So μ can be expressed as a convex combination of $\mu_1, \mu_2 \in \mathcal{M}(X,T,S)$ and is not an extreme point of $\mathcal{M}(X,T,S)$.

Suppose μ is $(T,S)-$ergodic, and $\mu = p\mu_1 + (1-p)\mu_2$ for some $\mu_1, \mu_2 \in \mathcal{M}(X,T,S)$ and $p \in (0,1)$. Then μ_1 is absolutely continuous with respect to μ, and the Radon-Nikodym derivative $\phi = d\mu_1/d\mu$ satisfies that $\phi(Tx) = \phi(x) = \phi(Sx)$, $\mu - a.e.$ So it must follow that $\phi(x) = 1$, $\mu - a.e$, because $\int_X \phi d\mu = 1$. Thus, $\mu = \mu_1$. Similarly, $\mu = \mu_2$. Hence μ is an extreme point of $\mathcal{M}(X,T,S)$. ∎

Proposition 1.4 shows the existence of $(T,S)-$ergodic measure and gives rise to the possibility of $(T,S)-$ergodic decomposition.

Notice if T and S commute, then so do T and TS.

Proposition 1.5. *Suppose T and S are homeomorphisms on X.*
i) $\mathcal{M}(X,T,TS) = \mathcal{M}(X,T,S)$.
ii) μ is $(T,S)-$ergodic iff μ is $(T,TS)-$ ergodic.

Proof. i) is clear. ii) follows from i) and Proposition 1.4. ∎

We denote by $\mathcal{E}(X,T)$ the set of ergodic measures under action T. Then by our notation, $\mathcal{M}(\mathcal{E}(X,T))$ is the set of Borel probability measures on $\mathcal{E}(X,T)$. It is known that for any $\nu \in \mathcal{M}(X,T)$, there exists a unique element $\pi \in \mathcal{M}(\mathcal{E}(X,T))$, such that ν has the ergodic decomposition ([W],Chapter 6)

$$\nu = \int_{\mathcal{E}(X,T)} \nu_e d\pi(\nu_e), \tag{Δ}$$

It means that $\forall\ \phi \in C(X)$,

$$\int_X \phi(x) d\nu(X) = \int_{\mathcal{E}(X,T)} (\int_X \phi(x) d\nu_e(x)) d\pi(\nu_e). \tag{$\Delta\Delta$}$$

In fact, given any $\pi \in \mathcal{M}(\mathcal{E}(X,T))$, this formula can determine a unique $T-$invariant measure ν on X as well. Thus we obtain a $1-1$ map $\tau : \mathcal{M}(X,T) \to \mathcal{M}(\mathcal{E}(X,T))$ defined by $\tau(\nu) = \pi$.

Now we give following remarks which may be helpful for understanding $(T,S)-$ergodic measures. Here we need assume that both T and S are homeomorphisms on a compact metric space X.

Remark 1.6. *S^* induces a map $S^{**} = (S^*)^*$ on the set $\mathcal{M}(\mathcal{E}(X,T))$ by $S^{**}(\nu) = \nu \circ (S^*)^{-1}$.*

Proof. S^* is invertible. By Proposition 1.1, $S^*(\mathcal{M}(X,T)) = \mathcal{M}(X,T)$. Since S^* is affine and $\mathcal{E}(X,T)$ is the set of extreme points of $\mathcal{M}(X,T)$, $S^*(\mathcal{E}(X,T)) = \mathcal{E}(X,T)$. Then the result follows. ∎

Remark 1.7.

i) $\mu \in \mathcal{M}(X,T)$ is $S-$invariant iff $\tau(\mu)$ is S^*-invariant. In other words, $\mu \in \mathrm{Fix}(S^*)$ iff $\tau(\mu) \in \mathrm{Fix}(S^{**})$.

ii) μ is $(T,S)-$ergodic iff $\tau(\mu)$ is ergodic with respect to S^*.

Proof. i) Denote $\pi = \tau(\mu)$. Since S^* is affine and maps $\mathcal{E}(X,T)$ to itself, we have that

$$S^*\mu = \int_{\mathcal{E}(X,T)} S^*\mu_e d\pi(\mu_e) = \int_{\mathcal{E}(X,T)} \mu_e d\pi((S^*)^{-1}\mu_e) = \int_{\mathcal{E}(X,T)} \mu_e d(S^{**}\pi)(\mu_e).$$

Comparing it with (Δ) we know that $S^*\mu = \mu$ iff $S^{**}\pi = \pi$.

ii) Notice that i) means $\tau(\mathcal{M}(X,T,S)) = \mathcal{M}(\mathcal{E}(X,T),S^*)$. Since τ is $1-1$ and affine, the extreme points of the two sets are corresponding under the action τ. ∎

§2. Lyapunov Exponents

From now on we back our discussion on the smooth dynamical systems. The suppositions on M, f, g and μ are as before. Recall that Γ_f is a subset of M such that $f\Gamma_f = \Gamma_f$, $\nu\Gamma_f = 1$, $\forall\, \nu \in \mathcal{M}(M,f)$, and for any $x \in \Gamma_f$,

$$\chi(x,u,f) = \lim_{n\to\infty} \frac{1}{n} \log \| Df_x^n u \|, \qquad \forall u \in T_xM.$$

Lemma 2.1. $\chi(gx, Dg_x u, f) = \chi(x,u,f)$.

Proof. There exists $C > 0$ such that $\forall x \in M, u \in T_xM$,

$$C^{-1} \| u \| \leq \| Dg_x u \| \leq C \| u \| .$$

Thus

$$C^{-1} \| Df_x^n u \| \leq \| Dg_{f^n x} Df_x^n u \| \leq C \| Df_x^n u \| .$$

So

$$\lim_{n\to\infty} \frac{1}{n} \log \| Df_{gx}^n Dg_x u \| = \lim_{n\to\infty} \frac{1}{n} \log \| Dg_{f^n x} Df_x^n u \| = \lim_{n\to\infty} \frac{1}{n} \log \| Df_x^n u \| . \qquad ∎$$

We know that the set Γ_f and the spectrum $\{\lambda_i(x,f), m_i(x,f)\}$ are $f-$invariant, and $Df_x(E_i(x,f)) = E_i(fx,f)$, $i = 1,\cdots,r(x,f)$. From above lemma we have the following.

Corollary 2.2.

i) The set Γ_f is $g-$invariant, i.e. $g\Gamma_f = \Gamma_f$.

ii) The spectrum $\{\lambda_i(x,f), m_i(x,f), \quad i = 1,\cdots,r(x,f)\}$ is $g-$invariant.

iii) $Dg_x(E_i(x,f)) = E_i(gx,f), \quad i = 1,\cdots,r(x,f)$.

Next proposition is a special case of Theorem A.

Proposition 2.3. *There exists a measurable set Γ_1 with $f\Gamma_1 = \Gamma_1 = g\Gamma_1$ and $\mu\Gamma_1 = 1$, $\forall\ \mu \in M(M,f,g)$, satisfying that for all $x \in \Gamma_1$, there is a decomposition of tangent space into*

$$T_xM = \bigoplus_{i=1}^{r(x,f)} \bigoplus_{j=1}^{r(x,g)} E_{ij}(x)$$

such that if $E_{ij}(x) \neq \{0\}$, then $\forall\ 0 \neq u \in E_{ij}(x)$,

$$\lim_{n\to\infty} \frac{1}{n} \log \|Df_x^n u\| = \lambda_i(x,f), \quad \lim_{n\to\infty} \frac{1}{n} \log \|Dg_x^n u\| = \lambda_j(x,f),$$

and

$$\lim_{n\to\infty} \frac{1}{n} \log |\sin(E_{i_1j_1}(f^n x), E_{i_2j_2}(f^n x))| = 0, \quad \lim_{n\to\infty} \frac{1}{n} \log |\sin(E_{i_1j_1}(g^n x), E_{i_2j_2}(g^n x))| = 0.$$

Moreover,the following invariant properties hold.

i) $Df_xE_{ij}(x) = E_{ij}(fx)$, $Dg_xE_{ij}(x) = E_{ij}(gx)$.

ii) $\lambda_i(fx,f) = \lambda_i(x,f) = \lambda_i(gx,f)$, $\lambda_j(fx,g) = \lambda_j(x,g) = \lambda_j(gx,g)$.

Proof. For any point $x \in \Gamma_f$, let $T_xM = E_i(x,f) \oplus \cdots \oplus E_{r(x,f)}(x,f)$ be the decomposition of tangent space for diffeomorphism f. By Corollary 2.2, $Dg_x(E_i(x,f)) = E_i(gx,f)$. Restricted on $\{E_i(x,f)\}$, $\{Dg_x^n\}$ is a cocycle on M with respect to g (see [Ru]), where we take $E_i(x,f) = \{0\}$ if $i > r(x,f)$ or x is not in Γ_f. Now we use the Multiplicative Ergodic Theorem for each i to get a subset $\Gamma^{(i)} \subset \Gamma$, such that $\forall\ x \in \Gamma^{(i)}$, after relabelling the subscript, if necessary, $E_{ij}(x)$ has desired properties. Since for each i, $\mu\Gamma^{(i)} = 1$, $\forall\ \mu \in M(M,f,g)$, and by Corollary 2.2.i), $f\Gamma^{(i)} = \Gamma^{(i)}$, we can take $\Gamma_1 = \bigcap_{i=1}^{r(x,f)} \Gamma^{(i)}$. ∎

The Proof of Theorem A.

First we claim that $\forall\ s,t \in \mathbb{Z}$, $i = 1,\cdots,r(x,f)$, $j = 1,\cdots,r(x,g)$, the set

$$A_\gamma = \{x:\ \exists u_x \in E_{ij}(x),\ s.t.\ \chi(x,u_x,f^sg^t) - s\lambda_i(x,f) - t\lambda_j(x,g) > 4\gamma\}$$

satisfies $\mu A_\gamma = 0$ for all $\mu \in \mathcal{M}(M,f,g)$.

Suppose it is not true. Then there exists a $\mu \in \mathcal{M}(M,f,g)$ with $\mu A_\gamma > 0$. Choose $l > 0$ such that the sets

$$A' = \{x \in A_\gamma : \|D(f^sg^t)_x^n u_x\| \geq l^{-1}\|u_x\| \exp n(\chi(x,u_x,f^sg^t) - \gamma),\ \forall n \geq 0\},$$

$$A'' = \{x \in A_\gamma : \|Dg_x^{tn} u\| \leq l\|u\| \exp n(t\lambda_j(x,g) + \gamma),\ \forall u \in E_{ij}(x),\ n \geq 0\}$$

have measure larger than $\frac{1}{2}\mu A_\gamma$. Then $A' \cap A'' \neq \emptyset$. By Poincaré Recurrence Theorem we can take $x \in A' \cap A''$ such that there exists a sufficient large integer $n > \frac{2\log l}{\gamma}$ with $f^{sn}x \in A' \cap A''$ and

$$\|Df_x^{sn} u\| \leq \|u\| \exp n(s\lambda_i(x,f) + \gamma), \qquad \forall u \in E_{ij}(x).$$

Since $Df_x^{sn}u \in E_{ij}(f^{sn}x)$ and $f^{sn}x \in A''$,

$$\begin{aligned}\|D(f^sg^t)_x^n u\| &= \|Dg_{f^{sn}x}^{tn} Df_x^{sn} u\| \leq l\|Df_x^{sn}u\| \exp n(t\lambda_j(f^{sn}x,g) + \gamma)\\ &\leq l\|u\| \exp n(s\lambda_i(x,f) + t\lambda_j(x,g) + 2\gamma)\\ &< l^{-1}\|u\| \exp n(\chi(x,u,f^sg^t) - \gamma), \qquad \forall u \in E_{ij}(x).\end{aligned}$$

In particular, take $u = u_x$, then

$$\|D(f^s g^t)^n_x u_x\| < l^{-1}\|u_x\| \exp n(\chi(x, u_x, f^s g^t) - \gamma).$$

This contradicts the fact $x \in A'$.

Similar claim for the set

$$B_\gamma = \{x : \ \exists u_x \in E_{ij}(x), \ s.t. \ \chi(x, u_x, f^s g^t) - s\lambda_i(x, f) - t\lambda_j(x, g) < 4\gamma\}$$

is also true. It is easy to see by the claims that for any $\mu \in \mathcal{M}(M, f, g)$, $\mu - a.e.x \in M$,$\forall \ s, t \in \mathbb{Z}$, $\ i = 1, \cdots, r(x, f)$, $j = 1, \cdots, r(x, g)$, if $E_{ij}(x) \neq \{0\}$, then $\forall \ 0 \neq u \in E_{ij}(x)$,

$$\chi(x, u, f^s g^t) = \lim_{n\to\infty} \frac{1}{n} \log \|D(f^s g^t)^n_x u\| = s\lambda_i(x, f) + t\lambda_j(x, g).$$

Using the same idea, with some modification, we can prove the result concerning the equality

$$\lim_{n\to\infty} \frac{1}{n} \log \left|\sin(E_{i_1 j_1}((f^s g^t)^n x), E_{i_2 j_2}((f^s g^t)^n x))\right| = 0.$$

The rest of the results of the theorem follow directly from Proposition 2.3. ∎

§3. A Version of Pesin's Theory

In this section we give a version of Pesin's theory in the case of commuting diffeomorphisms. The main result is stated in Proposition 3.6.

Lemma 3.1. *Let $A(x)$ be a positive measurable function on Γ such that there exist positive measurable functions $P_1(x)$ and $P_2(x)$ on Γ satisfying that for all $x \in \Gamma$,*

$$P_1(x)e^{-6(|n|+|k|)\gamma} \leq A((f^n g^k)x) \leq P_2(x)e^{6(|n|+|k|)\gamma}, \qquad \forall \ u, k \in \mathbb{Z}.$$

Then a measurable function $C : \Gamma \to [1, \infty)$ can be found such that, $\forall \ x \in \Gamma$,

$$C(x)^{-1} \leq A(x) \leq C(x),$$

and

$$C(f^{\pm 1}x) \leq C(x)e^{8\gamma}, \qquad C(g^{\pm 1}x) \leq C(x)e^{8\gamma}.$$

Proof. For any $x \in \Gamma$, except for finite number of pairs (n, k),

$$A((f^n g^k)x)e^{-8(|n|+|k|)\gamma} \leq 1, \qquad A((f^n g^k)x)^{-1}e^{-8(|n|+|k1|)\gamma} \leq 1.$$

Thus,

$$C(x) = \max\{1, \ A(f^n g^k x)e^{-8(|n|+|k|)\gamma}, \ A(f^n g^k x)^{-1}e^{-8(|n|+|k|)\gamma}, \quad \forall \ n, k \in \mathbb{Z}\}$$

is a required function. ∎

Lemma 3.2. *For any $\gamma > 0$, there exists a measurable function $Q : \Gamma \to [1, \infty)$ such that $\forall\ n, k \in \mathbb{Z}$, $0 \neq u \in E_{ij}(x)$, $x \in \Gamma$, $i = 1, \cdots, r(x, f)$, $j = 1, \cdots, r(x, g)$,*

$$\begin{aligned} Q(x)^{-1}\|u\| e^{n\lambda_i(x,f)+k\lambda_j(x,g)-3(|n|+|k|)\gamma} &\leq \|D(f^n g^k)_x u\| \\ &\leq Q(x)\|u\| e^{n\lambda_i(x,f)+k\lambda_j(x,g)+3(|n|+|k|)\gamma}. \end{aligned}$$

Proof. In the proof we always assume that $\|u\| = 1$.

We assert that for each $x \in \Gamma$, if $|n|$ or $|k|$ is sufficiently large, then

$$e^{-3(|n|+|k|)\gamma} \leq \|D(f^n g^k)_x u\| e^{-n\lambda_i(x,f)-k\lambda_j(x,g)} \leq e^{3(|n|+|k|)\gamma}.$$

First we suppose $n \geq k \geq 0$.

Take $l = l(x) > 0$, such that $\forall\ y \in \bigcup_{n,k\in\mathbb{Z}} (f^n g^k)x$, $v \in E_{ij}(y)$, $i = 1, \cdots, r(x, f)$, $i = 1, \cdots, r(x, g)$,

$$\|Df_y v\| \leq \|v\| e^{\lambda_i(x,f)+l\gamma}, \qquad \|Dg_y v\| \leq \|v\| e^{\lambda_j(x,g)+l\gamma}.$$

By Theorem A, $\forall\ 0 \leq t \leq l$, $u \in E_{ij}(x)$,

$$\lim_{s\to\infty} \frac{1}{s} \log \|D(f^l g^t)_x^s u\| = l\lambda_i(x, f) + t\lambda_j(x, g).$$

We can choose a positive integer $s_0 > l$, such that $\forall s > s_0$, $l \geq t \geq 0$,

$$e^{sl\lambda_i(x,f)+st\lambda_j(x,g)-(sl+st)\gamma} \leq \|D(f^l g^t)_x^s u\| \leq e^{sl\lambda_i(x,f)+st\lambda_j(x,g)+(sl+st)\gamma}.$$

Denote $N_I = ls_0$.

Take $n \geq N_I$. For $0 \leq k \leq n$, we can write $n = sl + p, k = ts + q$, where $0 \leq p < l$, $0 \leq q < s$. Notice that $\lambda_i(x, f)$ and $\lambda_j(x, g)$ are both f and $g-$invariant, we have

$$\begin{aligned} \|D(f^n g^k)_x u\| &= \|Df^p \circ Dg^q \circ D(f^l g^t)_x^s u\| \\ &\leq e^{p(\lambda_i(x,f)+l\gamma)} \cdot e^{q(\lambda_j(x,g)+l\gamma)} \cdot e^{sl\lambda_i(x,f)+st\lambda_j(x,g)+(sl+st)\gamma} \\ &= e^{n\lambda_i(x,f)+k\lambda_j(x,g)+(pl+ql+sl+st)\gamma} \\ &\leq e^{n\lambda_i(x,f)+k\lambda_j(x,g)+3(n+k)\gamma}. \end{aligned}$$

Also, we can write $n = sl - p, k = ts - q$, where $0 \leq p < l$, $0 \leq q < s$. Then

$$\begin{aligned} \|D(f^l g^t)_x^s u\| &= \|Df^p \circ Dg^q \circ D(f^n g^k)_x u\| \\ &\leq e^{p(\lambda_i(x,f)+l\gamma)} \cdot e^{q(\lambda_j(x,g)+l\gamma)} \|D(f^n g^k)_x u\|. \end{aligned}$$

So

$$\begin{aligned} \|D(f^n g^k)_x u\| &\geq e^{-p(\lambda_i(x,f)+l\gamma)} \cdot e^{-q(\lambda_j(x,g)+l\gamma)} \cdot e^{sl\lambda_i(x,f)+st\lambda_j(x,g)-(sl+st)\gamma} \\ &= e^{n\lambda_i(x,f)+k\lambda_j(x,g)-(pl+ql+sl+st)\gamma} \\ &\geq e^{n\lambda_i(x,f)+k\lambda_j(x,g)-3(n+k)\gamma}. \end{aligned}$$

These inequalities show that in the case of $n \geq k \geq 0$, our assertion is true if $n \geq N_I$. It is also true for the case $k \geq n \geq 0$ because we can find $K_I > 0$ similarly such that the inequality holds if $k \geq K_I$. Since the iterations of f and g in positive and negative directions are symmetrical, the assertion is true if one or two of n and k are negative.

Now we know that

$$\begin{aligned} Q(x) = \max\{1, &\|D(f^n g^k)_x u\|^{-1} e^{n\lambda_i(x,f)+k\lambda_j(x,g)-3(|n|+|k|)\gamma}, \\ &\|D(f^n g^k)_x u\| e^{-n\lambda_i(x,f)-k\lambda_j(x,g)-3(|n|+|k|)\gamma},\ \forall n,k \in \mathbb{Z}, u \in E_{ij}(x), \|u\| = 1\} \end{aligned}$$

is a required function. ∎

Remark 3.3. *In the proof of Lemma 3.2, $Q(x)$ is chosen to be the minimal function satisfying our requirement, i.e. $\forall\ x \in \Gamma$,*

$$\begin{aligned} Q(x) = \inf\{q \geq 1 :\ q^{-1} e^{-3(|n|+|k|)\gamma} \leq \|D(f^n g^k)_x u\| e^{-n\lambda_i(x,f)-k\lambda_j(x,g)} \leq q e^{3(|n|+|k|)\gamma} \\ \forall n,k \in \mathbb{Z}, u \in E_{ij}(x)\}. \end{aligned}$$

Lemma 3.4. *For any $\gamma > 0$, there exists a measurable function $R : \Gamma \to [1,\infty)$, such that $\forall\ n,k \in \mathbb{Z}$, $i_1, i_2 = 1, \cdots, r(x,f)$, $j_1, j_2 = 1, \cdots, r(x,g)$, $(i_1,j_1) \neq (i_2,j_2)$,*

$$\left|\sin\Big(E_{i_1 j_1}((f^n g^k)x), E_{i_2 j_2}((f^n g^k)x)\Big)\right| \geq R(x)^{-1} e^{-3(|n|+|k|)\gamma}.$$

Proof. The method is similar as in the proof for the left inequality of Lemma 3.2 if we use

$$\max\Big\{\frac{|\sin(E_{i_1 j_1}(fx), E_{i_2 j_2}(fx))|}{|\sin(E_{i_1 j_1}(x), E_{i_2 j_2}(x))|}\Big\}, \qquad \max\Big\{\frac{|\sin(E_{i_1 j_1}(gx), E_{i_2 j_2}(gx))|}{|\sin(E_{i_1 j_1}(x), E_{i_2 j_2}(x))|}\Big\}$$

instead of Df_x and Dg_x respectively, where the maximums run over $i_1, i_2 = 1, \cdots, r(x,f)$, $j_1, j_2 = 1, \cdots, r(x,g)$, $(i_1,j_1) \neq (i_2,j_2)$. ∎

Lemma 3.5. *The function $Q(x)$ determined by the proof of Lemma 3.2 satisfies that*

$$Q(f^s g^t x) \leq Q(x)^2 e^{6(|s|+|t|)\gamma}, \quad \forall\ s,t \in \mathbb{Z}.$$

Proof. Take $s,t \in \mathbb{Z}$, $\forall\ u \in E_{ij}(f^t g^t x)$, where $i = 1, \cdots, r(x,f)$, $j = 1, \cdots, r(x,g)$. We can choose $u' \in E_{ij}(x)$ with $u = D(f^s g^t)_x u'$. By Lemma 3.2,

$$Q(x)^{-1} \|u'\| e^{s\lambda_i(x,f)+t\lambda_j(x,g)-3(|s|+|t|)\gamma} \leq \|u\| \leq Q(x) \|u'\| e^{s\lambda_i(x,f)+t\lambda_j(x,g)+3(|s|+|t|)\gamma}.$$

Since $\forall\ n,k \in \mathbb{Z}$, $D(f^n g^k)_{f^s g^t x} u = D(f^{n+s} g^{k+t})_x u'$,

$$\begin{aligned} &Q(x)^{-1} \|u'\| e^{(n+s)\lambda_i(x,f)+(k+t)\lambda_j(x,g)-3(|n+s|+|k+t|)\gamma} \\ &\leq \|D(f^n g^k)_{f^s g^t x} u\| \leq Q(x) \|u'\| e^{(n+s)\lambda_i(x,f)+(k+t)\lambda_j(x,g)+3(|n+s|+|k+t|)\gamma}. \end{aligned}$$

From these inequalities we have that

$$
\begin{aligned}
&Q(x)^{-2}\|u\|e^{n\lambda_i(x,f)+k\lambda_j(x,g)-3(|n|+|k|)\gamma-6(|s|+|t|)\gamma} \\
\le\ &\|D(f^n g^k)_{f^s g^t x}u\| \le Q(x)^2\|u\|e^{n\lambda_i(x,f)+k\lambda_j(x,g)+3(|n|+|k|)\gamma+6(|s|+|t|)\gamma}.
\end{aligned}
$$

By Remark 3.3 the value of Q at $f^s g^t x$ must satisfy

$$Q(f^s g^t x) \le Q(x)^2 e^{6(|s|+|t|)\gamma}.$$

∎

Proposition 3.6. *Given M, f, g, Γ as in Theorem A, then for any $\gamma > 0$, there exists a measurable function $C : \Gamma \to [1,\infty)$ such that*

i) $\forall n, k \in \mathbb{Z}$, $u \in E_{ij}(x)$, $i = 1, \cdots, r(x,f)$, $j = 1, \cdots, r(x,g)$,

$$C(x)^{-1}\|u\|e^{n\lambda_i(x,f)+k\lambda_j(x,g)-(|n|+|k|)\gamma} \le \|D(f^n g^k)_x u\| \le C(x)\|u\|e^{n\lambda_i(x,f)+k\lambda_j(x,g)+(|n|+|k|)\gamma},$$

ii) $\forall i_1, i_2 = 1, \cdots, r(x,f)$, $j_1, j_2 = 1, \cdots, r(x,g)$, $(i_1, j_1) \ne (i_2, j_2)$,

$$|\sin(E_{i_1 j_1}(x), E_{i_2 j_2}(x))| \ge C(x)^{-1},$$

iii) $C(f^{\pm 1}x) \le C(x)e^{\gamma}, \quad C(g^{\pm 1}x) \le C(x)e^{\gamma}$.

Proof. In Lemma 3.5 we replace s,t by $-n$,$-k$, respectively, then replace x by $f^n g^k x$ to get

$$Q(x) \le Q(f^n g^k x)^2 e^{6(|n|+|k|)\gamma},$$

i.e.

$$Q(f^n g^k x) \ge \sqrt{Q(x)}e^{-3(|n|+|k|)\gamma}.$$

We let $P_1(x) = \sqrt{Q(x)}$ and $P_2(x) = Q(x)^2$. By Lemma 3.1 there exists a measurable function $C_1(x) > 0$ with $Q(x) \le C_1(x)$ and $C_1(f^{\pm 1}x) \le C_1(x)e^{8\gamma}$, $C_1(g^{\pm 1}x) \le C_1(x)e^{8\gamma}$.

Take

$$
\begin{aligned}
A(x) = \max\{&|\sin(E_{i_1 j_1}(x), E_{i_2 j_2}(x))| : \\
&i_1, i_2 = 1, \cdots, r(x,f), j_1, j_2 = 1, \cdots, r(x,g), (i_1, j_1) \ne (i_2, j_2)\}.
\end{aligned}
$$

By Lemma 3.4 and Lemma 3.1, there exists a measurable function $C_2(x) > 0$ such that $A(x) \ge C_2(x)^{-1}$ and $C_2(f^{\pm 1}x) \le C_2(x)e^{8\gamma}$, $C_2(g^{\pm 1}x) \le C_2(x)e^{8\gamma}$.

Now we use γ instead of 8γ and put $C(x) = \max\{C_1(x), C_2(x)\}$. Then $C(x)$ is a required function. ∎

§4. Lyapunov Charts

We have already had the decomposition of tangent space into subspaces corresponding Lyapunov exponents for both f and g. In this section we construct Lyapunov charts for the diffeomorphisms by the same method used in [LY]. For simplicity our discussion just concerns the difference and skips the rest.

Let $\langle\!\langle \cdot,\cdot \rangle\!\rangle$ be the inner product on T_xM given by the Riemannian structure and $\|\cdot\|$ be the induced norm. Let $\langle \cdot,\cdot \rangle$ and $|\cdot,\cdot|$ denote the usual inner product and norm in $I\!R^m$ respectively. Also, for $\rho > 0$, let $\tilde{B}(\rho)$ be the ball in $I\!R^m$ centered at origin of radius ρ.

Let

$$\lambda_+(x,f) = \min\{\lambda_i(x,f) : \ \lambda_i(x,f) > 0\}, \quad \lambda_-(x,f) = \max\{\lambda_i(x,f) : \ \lambda_i(x,f) < 0\},$$

$$\Delta\lambda(x,f) = \min\{\lambda_i(x,f) - \lambda_{i+1}(x,f), \quad i = 1,\cdots,r(x,f)-1\},$$

and define $\lambda_\pm(x,g)$, $\Delta\lambda(x,g)$ in similar way. Take

$$0 < \gamma = \gamma(x) \le \frac{1}{200m}\min\{\Delta\lambda(x,f), \Delta\lambda(x,g), \pm\lambda_\pm(x,f), \pm\lambda_\pm(x,g)\},$$

where $m = \dim M$. Next proposition shows the existence and properties of Lyapunov charts for f and g.

Proposition 4.1. *For the γ defined as above, there exists a measurable function $l : \Gamma \to [0,\infty)$ with $l(f^{\pm 1}x) \le l(x)e^\gamma$, $l(g^{\pm 1}x) \le l(x)e^\gamma$, and a set of embedings $\Phi_x : \tilde{B}(l(x)^{-1}) \to M$ at each point $x \in \Gamma$ such that the following holds.*

i) $\Phi_x(0) = x$, and the preimages $R_{ij}(x) = D\Phi_x(0)^{-1}(E_{ij}(x))$ of $E_{ij}(x)$ are mutually orthogonal in $I\!R^m$, where $i = 1,\cdots,r(x,f)$, $j = 1,\cdots,r(x,g)$.

ii) Let $\tilde{f}_x = \Phi_{fx}^{-1} \circ f \circ \Phi_x$ be the connecting map between the chart at x and the chart at fx. $\tilde{g}_x$, $\widetilde{(fg)}_x$ and their inverses are defined similarly. Then $\widetilde{(fg)}_x = \tilde{f}_{gx}\tilde{g}_x = \tilde{g}_{fx}\tilde{f}_x$ and $\widetilde{(fg)}_x^{-1} = \tilde{f}_{g^{-1}x}^{-1}\tilde{g}_x^{-1} = \tilde{g}_{f^{-1}x}^{-1}\tilde{f}_x^{-1}$.

iii) For any $1 \le i \le r(x,f)$, $l \le j \le r(x,g)$,and $u \in R_{ij}(x)$,

$$|u|e^{\lambda_i(x,f)-\gamma} \le |D\tilde{f}_x(0)\ u| \le |u|e^{\lambda_i(x,f)+\gamma},$$

$$|u|e^{\lambda_j(x,g)-\gamma} \le |D\tilde{g}_x(0)\ u| \le |u|e^{\lambda_j(x,g)+\gamma},$$

$$|u|e^{\lambda_i(x,f)+\lambda_j(x,g)-\gamma} \le |D\widetilde{(fg)}_x(0)\ u| \le |u|e^{\lambda_i(x,f)+\lambda_j(x,g)+\gamma}.$$

iv) Let $L(\Psi)$ be the Lipschitz constant of the function Ψ. Then for $F_x = \tilde{f}_x, \tilde{g}_x, \widetilde{(fg)}_x$ and their inverses,

$$L(F_x - DF_x(0)) \le \gamma, \qquad L(DF_x) \le l(x).$$

v) There exists a number $\lambda > 0$ depending on γ and the exponents such that $\forall x \in \Gamma$,

$$|\tilde{f}_x^{\pm 1}u| \le e^\lambda|u|, \quad |\tilde{g}_x^{\pm 1}u| \le e^\lambda|u|, \quad |\widetilde{(fg)}_x^{\pm 1}u| \le e^\lambda|u|, \quad \forall u \in \tilde{B}(e^{-\lambda-\gamma}l(x)^{-1}).$$

vi) For all $u,v \in \tilde{B}(l(x)^{-1})$, we have

$$K^{-1}d(\Phi_x u, \Phi_x v) \le |u-v| \le l(x)d(\Phi_x u, \Phi_x v),$$

for some universal constant K.

We shall refer to any system of local charts $\{\Phi_x : x \in \Gamma\}$ satisfying i) $-$ vi) as $(\gamma,l)-$charts for f and g. Obviously, if $\{\Phi_x : x \in \Gamma\}$ is a system of $(\gamma,l)-$charts for both f and g, then it is a system for either f or g as well.

The Proof of Proposition 4.1.

By Proposition 3.6 there exists a measurable function $C:\Gamma\to[1,\infty)$ such that for all $x\in\Gamma$, we have the following.

i) $\forall n,k\in\mathbb{Z}$, $u\in E_{ij}(x)$, $i=1,\cdots,r(x,f)$, $j=1,\cdots,r(x,g)$,

$$C(x)^{-1}\|u\|e^{n\lambda_i(x,f)+k\lambda_j(x,g)-(|n|+|k|)\gamma}\le\|D(f^ng^k)_xu\|\le C(x)\|u\|e^{n\lambda_i(x,f)+k\lambda_j(x,g)+(|n|+|k|)\gamma}.$$

ii) $\forall i_1,i_2=1,\cdots,r(x,f)$, $j_1,j_2=1,\cdots,r(x,g)$, $(i_1,j_1)\neq(i_2,j_2)$,

$$|\sin(E_{i_1j_1}(x),E_{i_2j_2}(x))|\ge C(x)^{-1}.$$

iii) $\quad C(f^{\pm1}x)\le C(x)e^{\gamma},\quad C(g^{\pm1}x)\le C(x)e^{\gamma}.$

We define a new inner product $\langle\!\langle\!\langle\cdot,\cdot\rangle\!\rangle\!\rangle$ on T_xM. First, for $u,v\in E_{ij}(x)$, let

$$\langle\!\langle\!\langle u,v\rangle\!\rangle\!\rangle=\sum_{n=-\infty}^{+\infty}\sum_{k=-\infty}^{+\infty}\frac{\langle\!\langle D(f^ng^k)_xu,\ D(f^ng^k)_xv\rangle\!\rangle}{\exp 2[n\lambda_i(x,f)+k\lambda_j(x,g)+2(|n|+|k|)\gamma]}.$$

Then we extend $\langle\!\langle\!\langle\cdot,\cdot\rangle\!\rangle\!\rangle$ to T_xM by demanding that all subspaces $\{E_{ij}(x)\}$ be mutually orthogonal with respect to $\langle\!\langle\!\langle\cdot,\cdot\rangle\!\rangle\!\rangle$. Let $|\!|\!|\cdot|\!|\!|$ be the corresponding norm. A calculation shows that $\forall\ 0\neq u\in E_{ij}(x)$,

$$\|u\|\le|\!|\!|u|\!|\!|\le C_0C(x)\|u\|,$$

where $C_0=(\sum\limits_{n=-\infty}^{+\infty}\sum\limits_{k=-\infty}^{+\infty}e^{-2(|n|+|k|)\gamma})^{\frac12}$, and

$$|\!|\!|u|\!|\!|e^{\lambda_i(x,f)-2\gamma}\le|\!|\!|Df_xu|\!|\!|\le|\!|\!|u|\!|\!|e^{\lambda_i(x,f)+2\gamma},$$

$$|\!|\!|u|\!|\!|e^{\lambda_j(x,g)-2\gamma}\le|\!|\!|Dg_xu|\!|\!|\le|\!|\!|u|\!|\!|e^{\lambda_j(x,g)+2\gamma},$$

and then

$$|\!|\!|u|\!|\!|e^{\lambda_i(x,f)+\lambda_j(x,g)-4\gamma}\le|\!|\!|Df_{gx}\cdot Dg_xu|\!|\!|\le|\!|\!|u|\!|\!|e^{\lambda_i(x,f)+\lambda_j(x,g)+4\gamma}.$$

For arbitrary $0\neq u\in T_xM$, we can write $u=\sum\limits_{i=1}^{r(x,f)}\sum\limits_{j=1}^{r(x,g)}u_{ij}$, where $u_{ij}\in E_{ij}(x)$. It is clear that

$$\|u\|\le\sum_i\sum_j\|u_{ij}\|\le\sum_i\sum_j|\!|\!|u_{ij}|\!|\!|\le m|\!|\!|u|\!|\!|,$$

because there are at most m different subspaces $E_{ij}(x)$ in T_xM. With similar argument in [LY, Appendix] we obtain that

$$\|u\|\ge\|u_{ij}\|C(x)^{-m+1},\qquad\forall i=1,\cdots,r(x,f),\ j=1,\cdots,r(x,g).$$

Therefore

$$|\!|\!|u|\!|\!|\le\sum_{i=1}^{r(x,f)}\sum_{j=1}^{r(x,g)}|\!|\!|u_{ij}|\!|\!|\le C_0C(x)\sum_{i=1}^{r(x,f)}\sum_{j=1}^{r(x,g)}\|u_{ij}\|\le mC_0C(x)^m\|u\|.$$

Define a linear map $L_x : T_xM \to I\!R^m$ such that

$$\langle L_x u,\, L_x v\rangle = \langle\!\langle\!\langle u,\, v\rangle\!\rangle\!\rangle, \qquad \forall u, v \in T_xM,$$

and let $\Phi_x = \exp_x \circ L_x^{-1}$, then i)$-$iii) hold, if we use γ instead of $\max\{m\gamma, 4\gamma\}$. To get iv)$-$vi) we take $l(x) = CC(x)^m$, where C is large enough and chosen in a way similar as in [LY, Appendix], and take $K = 2m$ for vi). The proof is finished. ∎

Corollary 4.2. *With above notation,* $\forall\ 0 < \epsilon < e^{-\gamma}$,

$$\tilde f_x^{\pm 1}\tilde B(\epsilon e^{-\lambda} l(x)^{-1}) \subset \tilde B(\epsilon l(fx)^{-1}), \quad \tilde g_x^{\pm 1}\tilde B(\epsilon e^{-\lambda} l(x)^{-1}) \subset \tilde B(\epsilon l(gx)^{-1}).$$

Proof. It is Proposition 4.1.v). ∎

Now we introduce a new norm $|\cdot|'$ on $I\!R^m$. For any $v \in I\!R^m$, we may write $v = \sum_{i=1}^{r(x,f)} \sum_{j=1}^{r(x,g)} v_{ij}$, where $v_{ij} \in R_{ij}$ as $R_{ij} \neq \{0\}$. Then let

$$|v|' = \max\{|v_{ij}|, \quad \forall i = 1, \cdots, r(x,f),\ j = 1, \cdots, r(x,g)\}.$$

It is clear that $\frac{1}{\sqrt{m}}|v| \le |v|' \le |v|$, where $m = \dim M$, and Proposition 4.1 still holds for $|\cdot|'$, but the universal constant $K = 2m$ may change to $K = 2m\sqrt{m}$. From here on, we use $|\cdot|'$ as the norm in $I\!R^m$ and still write it as $|\cdot|$.

Suppose $\{\Phi_x : x \in \Gamma\}$ is a system of $(\gamma, l)-$charts for f. For $x \in \Gamma$, let $R^u = L_x^{-1}E^u(x,f)$,$R^{sc} = L_x^{-1}E^{sc}(x,f)$ and so on. Then $I\!R^m = R^s \times R^c \times R^u$, or $I\!R^m = R^{sc} \times R^u$. By Proposition 4.1.ii), $D\Phi_x(0)$ takes R^u,R^c,R^s to $E^u(x,f)$,$E^c(x,f)$,$E^s(x,f)$ respectively. The $u-$coordinate of a point $v \in I\!R^m$ is denoted by v_u. Other notations such as v_c, v_s, v_{sc} are understood in obvious way. Clearly

$$|v| = \max\{|v_s|, |v_c|, |v_u|\}.$$

We should remember that the notations v_u,v_{sc} depend on the diffeomorphism, but we will not indicate it in our notation, because there is no ambiguity from context.

Corollary 4.3. *Let $x \in \Gamma$ and $0 < \epsilon < e^{-\lambda-\gamma}$. We have the following.*
i) If $v, v' \in B(\epsilon l(x)^{-1})$ and $|v - v'| = |v_u - v'_u|$, then

$$|\tilde f_x v - \tilde f_x v'| = |(\tilde f_x v)_u - (\tilde f_x v')_u| \ge e^{\lambda_+(x,f) - 2\gamma}|v - v'|.$$

ii) If $v, v' \in B(\epsilon l(x)^{-1})$ and $|v - v'| = |v_{sc} - v'_{sc}|$, then

$$|\tilde f_x^{-1} v - \tilde f_x^{-1} v'| = |(\tilde f_x^{-1} v)_{sc} - (\tilde f_x^{-1} v')_{sc}| \ge e^{-2\gamma}|v - v'|.$$

Proof. It follows from Proposition 4.1.iii)-v). ∎

§5. Unstable Manifold

Recall that the global unstable manifold of a diffeomorphism f at x is the set $w^u(x,f) = \{y \in M : \limsup_{n\to\infty} \frac{1}{n} \log d(f^{-n}x, f^{-n}y) < 0\}$.

Let $\{\Phi_x : x \in \Gamma\}$ be a system of $(\gamma, l)-$ charts for f. The local unstable manifold, denoted by $w^u_\alpha(x,f)$, of f at x associated with $\{\Phi_x\}$ and α is defined to be the component of $w^u(x,f) \cap \Phi_x R(\alpha l(x)^{-1})$ that contains x. The $\Phi_x^{-1}-$ image of this set in the $x-$chart is denoted by $W^u_\alpha(x,f)$.

The following result is well-known.

Proposition 5.1. *For $0 < \alpha \leq 1$ and $x \in \Gamma$, $W^u(x,f)$ is the graph of a function*

$$\psi_x^f : B^u(\alpha l(x)^{-1}) \longrightarrow B^{sc}(\alpha l(x)^{-1})$$

with $\psi_x^f(0) = 0$ and $\| D\psi_x^f \| \leq \frac{1}{3}$, where $B^u(\beta)$ and $B^{sc}(\beta)$ denote the balls centered at the origin of radius β in R^u and R^{sc} respectively.

The family of global unstable manifolds $\{w^u(x,f) : x \in \Gamma\}$ is $f-$invariant, that is, $fw^u(x,f) = w^u(fx,f)$. If f and g commute, we have the following.

Lemma 5.2. $fw^u(x,g) = w^u(fx,g)$.

Proof. There exists $C > 0$,such that $\forall x, y \in M$,

$$C^{-1}d(x,y) \leq d(fx,fy) \leq Cd(x,y).$$

Therefore

$$\begin{aligned} fw^u(x,g) &= \{y \in M : \limsup_{n\to\infty} \frac{1}{n} \log d(g^{-n}x, g^{-n}(f^{-1}y)\,) < 0\} \\ &= \{y \in M : \limsup_{n\to\infty} \frac{1}{n} \log d(g^{-n}(fx), g^{-n}y) < 0\} \\ &= w^u(fx,g). \end{aligned}$$

∎

Now we consider the case that $E^u(x,f) = E^u(x,g)$, $\mu - a.e.$ Let $\{\Phi_x : x \in \Gamma\}$ be a system of $(\gamma, l)-$charts for both f and g. Then by Proposition 5.1 there are two functions ψ_x^f and ψ_x^g defined on the balls of radius $\alpha l(x)^{-1}$ centered at origin in R^u, where $R^u = (D\Phi_x(0))^{-1}E^u(x,f) = (D\Phi_x(0))^{-1}E^u(x,g)$. The graphs of ψ_x^f and ψ_x^g are $W^u_\alpha(x,f)$ and $W^u_\alpha(x,g)$ respectively.

Lemma 5.3. *Suppose $E^u(x,f) = E^u(x,g)$, $\mu - a.e.$ and $0 < \alpha < e^{-\lambda-\gamma}$. Then $\forall\, x \in \Gamma$,*
i) $\tilde{f}_x^{-1} W^u_\alpha(x,f) \subset W^u_\alpha(f^{-1}x,f)$, $\quad \tilde{f}_x^{-1} W^u_\alpha(x,g) \subset W^u_\alpha(f^{-1}x,g)$;
ii) $\forall v \in W^u_\alpha(x,f)$ or $W^u_\alpha(x,g)$,

$$|\tilde{f}_x^{-n} v| = |(\tilde{f}_x^{-n} v)_u| \leq |v| \exp[-n(\lambda_+(x,f) - 2\gamma)].$$

Proof. We only need prove the result related to $W^u_\alpha(x,g)$. Take $v \in W^u_\alpha(x,g)$ arbitrary. Then $|v| \leq \alpha l(x)^{-1}$. By Proposition 4.1.v), $|\tilde{f}_x^{-1} v| \leq e^{-\gamma} l(x)^{-1} \leq l(f^{-1}x)^{-1}$, i.e.

$\tilde{f}_x^{-1}v \in B(l(f^{-1}x)^{-1})$. It is easy to know by Lemma 5.2 and Proposition 5.1 that $\tilde{f}_x^{-1}v \in W^u_{\alpha'}(f^{-1}x,g)$ for some $0<\alpha'<1$. Thus, Proposition 5.1 gives that $|\tilde{f}_x^{-1}v| = |(\tilde{f}_x^{-1}v)_u|$. By Corollary 4.3.i),

$$|v| = |\tilde{f}_{f^{-1}x}\tilde{f}_x^{-1}v| \geq e^{\lambda_+(x,f)-2\gamma}|\tilde{f}_x^{-1}v|,$$

i.e.

$$|\tilde{f}_x^{-1}v| \leq e^{-(\lambda_+(x,f)-2\gamma)}|v| \leq |v|e^{-\gamma} \leq \alpha l(f^{-1}x).$$

Thus, $|\tilde{f}_x^{-1}v| \in l(f^{-1}x)^{-1}$ and i) holds. Continuing this process we obtain ii). ∎

Proposition 5.4. *Suppose M is a C^∞ compact Riemannian manifold without boundary, $f,g \in \mathrm{Diff}^2(M)$ with $fg=gf$, and Γ is as in Theorem A. For any $x \in \Gamma$, if $E^u(x,f) = E^u(x,g)$, then $w^u(x,f) = w^u(x,g)$.*

Proof. We only need show that for some $0<\alpha<e^{-\lambda-\gamma}$, $W^u_\alpha(x,f) = W^u_\alpha(x,g)$.

Suppose it is not true. Then we can find $u \in W^u(x,f)$, $v \in W^u(x,g)$, such that

$$|u-v| = |u_{sc}-v_{sc}| > 0.$$

where u_{sc},v_{sc} are the $sc-$coordinates of u and v respectively. By Lemma 5.3, we have that

$$|(\tilde{f}_x^{-n}u)_u| = |\tilde{f}_x^{-n}u| \leq e^{-n(\lambda_+(x,f)-2\gamma)}|u| \leq \alpha e^{-n(\lambda_+(x,f)-2\gamma)},$$

$$|(\tilde{f}_x^{-n}v)_u| = |\tilde{f}_x^{-n}v| \leq e^{-n(\lambda_+(x,f)-2\gamma)}|v| \leq \alpha e^{-n(\lambda_+(x,f)-2\gamma)}$$

and $\tilde{f}_x^{-n}u$,$\tilde{f}_x^{-n}v \in B(\alpha l(f^{-n}x)^{-1})$ for any $n>0$. Applying Corollary 4.3.ii) repeatedly, we get

$$|(\tilde{f}_x^{-n}u)_{sc} - (\tilde{f}_x^{-n}v)_{sc}| \geq e^{-2n\gamma}|u-v|.$$

Without loss generality we may assume that $\lambda_+(x,f) \geq \lambda_+(x,g)$. Since Proposition 5.1 implies $|(\tilde{f}_x^{-n}u)_{sc}| \leq \frac{1}{3}|(\tilde{f}_x^{-n}u)_u|$, we get $\forall\ n \geq 0$,

$$\begin{aligned}
\frac{|(\tilde{f}_x^{-n}v)_{sc}|}{|(\tilde{f}_x^{-n}v)_u|} &\geq \frac{|(\tilde{f}_x^{-n}u)_{sc} - (\tilde{f}_x^{-n}v)_{sc}| - |(\tilde{f}_x^{-n}u)_{sc}|}{|(\tilde{f}_x^{-n}v)_u|} \\
&\geq \frac{e^{-2n\gamma}|u-v| - \frac{1}{3}\alpha e^{-n(\lambda_+(x,f)-2\gamma)}}{\alpha e^{-n(\lambda_+(x,g)-2\gamma)}} \\
&= \alpha^{-1}|u-v|e^{n(\lambda_+(x,g)-4\gamma)} - \frac{1}{3}e^{n(\lambda_+(x,g)-\lambda_+(x,f))}.
\end{aligned}$$

By our assumption, $\lambda_+(x,g)-\lambda_+(x,f) \leq 0$. Hence the right hand side in above inequality tends to infinite as $n \to \infty$. But by Proposition 5.1, $|(\tilde{f}_x^{-n}v)_{sc}| \leq \frac{1}{3}|(\tilde{f}_x^{-n}v)_u|$. This is a contradiction. ∎

§6. Local Entropies

Suppose $\nu \in \mathcal{M}(M,f)$, but not necessary in $\mathcal{M}(M,g)$. Let $B(x,\epsilon)$ be a closed ball in M centered at x of radius ϵ. We call the set

$$B_n(x,\epsilon,f) = \bigcap_{i=0}^{n} f^{-i}B(f^ix,\epsilon)$$

to be an $(n,\epsilon,f)-$ball of f at $x\in M$. The local entropy $h_\nu(x,f)$ of f at x is defined as (see [BK])

$$\begin{aligned} h_\nu(x,f) &= \lim_{\epsilon\to 0}\limsup_{n\to\infty} -\frac{1}{n}\log \nu B_n(x,\epsilon,f) \\ &= \lim_{\epsilon\to 0}\liminf_{n\to\infty} -\frac{1}{n}\log \nu B_n(x,\epsilon,f) \end{aligned}$$

which holds for $\nu - a.e.\ x\in M$, and satisfies that $h_\nu(fx,f)=h_\nu(x,f)$ and $\int h_\nu(x,f)d\nu(x) = h_\nu(f)$.

Recall $g^*\nu = \nu\circ g^{-1}$. Since ν is a fixed point of f^* and $f^*g^* = g^*f^*$, by Proposition 1.1, $g^*\nu$ is also a fixed point of f^*, i.e. $g^*\nu\in\mathcal{M}(M,f)$.

Lemma 6.1. *$\forall\ \nu\in\mathcal{M}(M,f)$, $h_{g^*\nu}(x,f) = h_\nu(g^{-1}x,f)$, $\nu - a.e.x\in M$. Therefore, $h_{g^*\nu}(f)=h_\nu(f)$.*

Proof. Since $g\in \mathrm{Diff}^2(M)$, there exists $C>1$, such that $\forall\ x\in M$, $\epsilon>0$,

$$B(g^{-1}x, C^{-1}\epsilon)\subset g^{-1}B(x,\epsilon)\subset B(g^{-1}x, C\epsilon).$$

Hence

$$B_n(g^{-1}x, C^{-1}\epsilon, f)\subset g^{-1}B_n(x,\epsilon,f)\subset B_n(g^{-1}x, C\epsilon, f).$$

So, by the definition of local entropy,

$$\begin{aligned} h_{g^*\nu}(x,f) &= \lim_{\epsilon\to 0}\limsup_{n\to\infty} \frac{1}{n}\log \nu(g^{-1}B_n(x,\epsilon,f)) \\ &= \lim_{\epsilon\to 0}\limsup_{n\to\infty} \frac{1}{n}\log \nu B_n(g^{-1}x,\epsilon,f) = h_\nu(g^{-1}x,f). \end{aligned}$$

This is the first result. Also we have

$$h_{g^*\nu}(f) = \int h_{g^*\nu}(x,f)d(g^*\nu)(x) = \int h_\nu(g^{-1}x,f)d\nu(g^{-1}x) = \int h_\nu(y,f)d\nu(y) = h_\nu(f). \quad \blacksquare$$

Since $\forall\ \mu\in\mathcal{M}(M,f,g)$, $g^*\mu=\mu$, the following fact can be induced directly from Lemma 6.1.

Corollary 6.2. *$h_\mu(x,f)$ is both f and $g-$invariant. Consequently, if μ is $(f,g)-$ergodic, then $h_\mu(x,f)=h_\mu(f)$, $\mu - a.e.\ x\in\Gamma$.*

Suppose a measure μ is given. For $\delta\in(0,1)$, we denote by $N_n(\epsilon,\delta,f)$ the minimal number of $(n,\epsilon,f)-$balls covering a set of the measure more than or equal to $1-\delta$. A. Katok has proved (see [K]) that if μ is an ergodic measure for f, then for every $\delta\in(0,1)$, $h_\mu(f) = \lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log N_n(\epsilon,\delta,f)$. Now we have same result for the measure μ which is $(f,g)-$ergodic.

Proposition 6.3. *If μ is an $(f,g)-$ergodic measure on M, then $\forall\ \delta\in(0,1)$,*

$$h_\mu(f) = \lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log N_n(\epsilon,\delta,f).$$

Proof. Take $\gamma > 0$ arbitrary.

Since

$$h_\mu(x,f) = \lim_{\epsilon\to 0}\liminf_{n\to\infty} -\frac{1}{n}\log \mu B_n(x,\epsilon,f), \qquad \mu - a.e.\ x \in M,$$

we can choose an $\epsilon > 0$ and an $n_0 > 0$ such that $\forall\ n > n_0$, the set

$$A_n = \{x \in M :\ \mu B_n(x,2\epsilon,f) \le \exp -n(h_\mu(f)-\gamma)\ \}$$

has measure larger than or equal to $\frac{1}{2}(1+\delta)$.

Let A'_n be a set that can be covered by $N_n(\epsilon,\delta,f)$ $(n,\epsilon,f)-$balls. $A'_n \cap A_n \neq \emptyset$ because $\mu(A'_n \cap A_n) \ge \frac{1}{2}(1-\delta)$. Thus $N_n(\epsilon,\delta,f)$ $(n,\epsilon,f)-$balls can cover $A'_n \cap A_n$. On the other hand, to cover $A'_n \cap A_n$ by $(n,2\epsilon,f)-$balls centered at points in $A'_n \cap A_n$, the number of such balls can not be less than $\frac{1-\delta}{2}\exp n(h_\mu(f)-\gamma)$. Since each $(n,\epsilon,f)-$ball whose intersection with set $A'_n \cap A_n$ is nonempty must be contained in an $(n,2\epsilon,f)-$ball centered at a point in $A'_n \cap A_n$, we have

$$N_n(\epsilon,\delta,f) \ge \frac{1-\delta}{2}\exp n(h_\mu(f)-\gamma).$$

It is true for any $n > n_0$. Hence

$$\limsup_{n\to\infty}\frac{1}{n}\log N_n(\epsilon,\delta,f) \ge h_\mu(f)-\gamma,$$

and therefore

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log N_n(\epsilon,\delta,f) \ge h_\mu(f).$$

The inequality in another direction can be obtained in similar way. ∎

In our discussion we only need the inequality with the proved direction.

Suppose $\rho : M \to \mathbb{R}_+$ is a measurable function. Define an $(n,\rho,f)-$ball at $x \in M$ by

$$B_n(x,\rho,f) = \bigcap_{i=0}^{n} f^{-i}B(f^i x, \rho(f^i x)\ ).$$

Proposition 6.4. *Let $\{\rho_\epsilon :\ \epsilon > 0\}$ be a family of functions on M satisfying that*
i) $0 < \rho_\epsilon \le \epsilon, \qquad \forall\ x \in M,$
ii) $\int \log\rho_\epsilon d\mu < \infty, \qquad \forall\ \epsilon > 0,$
iii) ρ_ϵ monotonously decreases as $\epsilon \to 0$.
Then

$$h_\mu(x,f) = \lim_{\epsilon\to 0}\limsup_{n\to\infty} -\frac{1}{n}\log \mu B(x,\rho_\epsilon,f), \qquad \mu - a.e.\ x \in \Gamma.$$

Proof. Clearly $B_n(x,\rho_\epsilon,f) \subset B_n(x,\epsilon,f)$. Hence

$$h_\mu(x,f) \le \lim_{\epsilon\to 0}\limsup_{n\to\infty} -\frac{1}{n}\log \mu B_n(x,\rho_\epsilon,f).$$

By the results of Mañé [M], Brin and Katok [BK],

$$h_\mu(f) \ge \int \lim_{\epsilon\to 0}\limsup_{n\to\infty} -\frac{1}{n}\log \mu B_n(x,\rho_\epsilon,f)d\mu \ge \int h_\mu(x,f)d\mu = h_\mu(f).$$

So the equalities hold everywhere and the result follows. ∎

§7. The Subadditivity of Entropies of Commuting Diffeomorphisms

We devote this section to the proof of the first part of Theorem B. The method we use here is estimating the number of $(n,\epsilon)-$balls which cover the set of measure more than or equal to $1-\delta$ for some constant $\delta\in(0,1)$.

Let $\{\Phi_x : x\in\Gamma\}$ be a system of $(\gamma,l)-$charts for both f and g. Recall that $\tilde{B}(\rho)$ denote the ball in $I\!R^m$ centered at the origin of radius ρ, and the maps Φ_x and $\tilde{f}_x$ are defined on $\tilde{B}(l(x)^{-1})$. Sometimes we will omit the subscript x.

For $l>0$, let $\Gamma_l=\{x\in\Gamma : l(x)<l\}$.

Lemma 7.1. *Suppose $y\in\Gamma_l\cap f^{-n}\Gamma_l$ and $0<\epsilon e^{-2n\gamma}<l^{-1}$. Let $E\subset I\!R^m$.*
i) If $E\subset\tilde{B}(\epsilon e^{-2n\gamma})$, $\tilde{f}^n_yE\subset\tilde{B}(\epsilon)$,and $\tilde{f}^i_yE\subset\tilde{B}(l(f^iy)^{-1})$, $\forall i=1,\cdots,n$, then

$$\tilde{f}^i_yE\subset\tilde{B}(\epsilon e^{-2(n-i)\gamma}),\quad\forall i=0,1,\cdots,n.$$

ii) If E,$\tilde{f}^n_yE\subset\tilde{B}(\epsilon e^{-2n\gamma})$, and $\tilde{f}^i_yE\subset\tilde{B}(l(f^iy)^{-1})$, $\forall i=1,\cdots,n-1$, then

$$\tilde{f}^i_yE\subset\tilde{B}(\epsilon e^{-2\max\{n-i,i\}\gamma}),\quad\forall i=0,1,\cdots,n.$$

Proof. i) Suppose there is $v\in E$ with $|\tilde{f}^i_yv|>\epsilon e^{-2(n-i)\gamma}$ for some $i\in(0,n)$. If $|\tilde{f}^i_yv|=|(\tilde{f}^i_yv)_u|$, then by Corollary 4.3.i),

$$|\tilde{f}^n_yv|=|\tilde{f}^{n-i}(\tilde{f}^i_yv)|\geq e^{(n-i)(\lambda_+(x,f)-2\gamma)}|\tilde{f}^i_yv|>e^{2(n-i)\gamma}\cdot\epsilon e^{-2(n-i)\gamma}=\epsilon.$$

If $|\tilde{f}^i_yv|=|(\tilde{f}^i_yv)_{cs}|$, then by Corollary 4.3.ii),

$$|v|=|\tilde{f}^{-i}(\tilde{f}^i_yv)|\geq e^{-2i\gamma}|\tilde{f}^i_yv|>e^{-2i\gamma}\cdot\epsilon e^{-2(n-i)\gamma}=\epsilon e^{-2n\gamma}.$$

Both cases are impossible.

ii) By part i), $\tilde{f}^i_yE\subset\tilde{B}(\epsilon e^{-2(n-i)\gamma})$. Again, using part i) on the set $\tilde{f}^n_yE$ for f^{-1}, we get $\tilde{f}^i_yE\subset\tilde{B}(\epsilon e^{-2i\gamma})$. The proof is finished by combining the inclusions. ∎

Take $l>0$ such that $\mu\Gamma_l>0$. For any $x\in\Gamma_l$, let $\tau_f(x)$ be the smallest positive integer k such that $f^kx\in\Gamma_l$. By Poincaré Recurrence Theorem for $\mu-a.e$ $x\in\Gamma_l$, $\tau_f(x)<\infty$. We extend $\tau_f(x)$ to M by putting $\tau_f(x)=0$ if $x\in M\backslash\Gamma_l$.

For any $\epsilon>0$, define a function $\rho_{\epsilon,f}:M\to I\!R_+$ by

$$\rho_{\epsilon,f}(x)=\min\{\epsilon,l^{-2}e^{-(\lambda+\gamma)\tau_f(x)}\}.$$

Now $\log\rho_{\epsilon,f}$ is integrable for any $\epsilon>0$ because $\int_{\Gamma_l}\tau_f(x)\leq 1$. So the family of functions $\{\rho_{\epsilon,f}:\ \epsilon>0\ \}$ satisfies the conditions in Proposition 6.4.

Lemma 7.2. *Let $0<\epsilon<l^{-2}$ and $\rho_{\epsilon,f}$ be defined as above. If $y\in\Gamma_l\cap f^{-n}\Gamma_l$, then*

$$\tilde{f}^i_y\Phi_y^{-1}\left[B_n(y,\rho_{\epsilon,f},f)\cap B(y,\epsilon e^{-2n\gamma})\right]\subset\tilde{B}(\epsilon le^{-2(n-i)\gamma}),\quad\forall i=0,\cdots,n.$$

Proof. Denote
$$E=\Phi_y^{-1}\big[B_n(y,\rho_{\epsilon,f},f)\cap B(y,\epsilon e^{-2n\gamma})\big].$$
Take $v\in E$ arbitrary and let $z=\Phi_y v$.

Clearly $d(y,z)\le \epsilon e^{-2n\gamma}$, $d(f^n y,f^n z)\le \rho_{\epsilon,f}(f^n y)\le\epsilon$. Therefore by Proposition 4.1.vi),
$$|v|=|\Phi_y^{-1}(z)|\le \epsilon l e^{-2n\gamma},\quad |\tilde{f}_y^n v|=|\Phi_{f^n y}^{-1}(f^n z)|\le \epsilon l.$$

For $i=1,\cdots,n$, we have that
$$d(f^i y,f^i z)\le \rho_{\epsilon,f}(f^i y)=\min\{\epsilon, l^{-2}e^{-(\lambda+\gamma)\tau_f(f^i y)}\}.$$
If $f^i y\in\Gamma_l$, then $d(f^i y,f^i z)\le l^{-2}\le l(f^i y)^{-2}$ and therefore $|\tilde{f}_y^i v|=|\Phi^{-1}(f^i z)|\le l(f^i y)^{-1}$. Suppose $f^i y\notin\Gamma_l$. Let $j<i$ be the largest integer such that $f^j y\in\Gamma_l$, then $\tau_f(f^j y)\ge i-j$ and $d(f^j y,f^j z)\le l^{-2}e^{-(\lambda+\gamma)(i-j)}\le l(f^j y)^{-2}e^{-(\lambda+\gamma)(i-j)}$. So we have
$$|\tilde{f}_y^j v|=|\Phi_{f^j y}^{-1}(f^j z)|\le l(f^j y)^{-1}e^{-(\lambda+\gamma)(i-j)}.$$
Thus, by Proposition 4.1.v),
$$|\tilde{f}_y^i v|=|\tilde{f}^{i-j}(\tilde{f}_y^j v)|\le e^{(i-j)\lambda}|\tilde{f}_y^j v|\le l(f^j y)^{-1}e^{-(i-j)\gamma}\le l(f^i y)^{-1}.$$
Now we know that E satisfies the conditions of Lemma 7.1.i) and our result follows. ∎

Lemma 7.3. *Suppose $0<\epsilon<l^{-2}e^{-\lambda-2\gamma}$. For $y\in\Gamma_l\cap f^{-n}\Gamma_l\cap(fg)^{-n}\Gamma_l$, if we set*
$$\Delta=B_n(y,\rho_{\epsilon,f},f)\cap B(y,\epsilon e^{-2n\gamma})\bigcap(fg)^{-n}\Big[B_n((fg)^n y,\rho_{\epsilon,g^{-1}},g^{-1})\cap B((fg)^n y,\epsilon e^{-2n\gamma})\Big],$$
then
$$\Delta\subset B_n(y,\epsilon l K,fg),$$
where K is as in Proposition 4.1.vi).

Proof. Let
$$E=\Phi_y^{-1}\Delta.$$
By Lemma 7.2, $\forall\ k=0,1,\cdots,n$,
$$\tilde{f}_y^k E\subset \tilde{f}_y^k\Phi_y^{-1}[B_n(y,\rho_{\epsilon,f},f)\cap B(y,\epsilon e^{-2n\gamma})]\subset\tilde{B}(\epsilon l e^{-2(n-k)\gamma}).$$
Similarly, since $(\widetilde{fg})^n\tilde{g}^{-k}\Phi_y^{-1}=\tilde{g}^{-k}\Phi_{(fg)^n y}^{-1}(fg)^n$, $\forall\ k=0,1,\cdots,n$,
$$(\widetilde{fg})^n\tilde{g}_y^{-k}E\subset\tilde{g}^{-k}\Phi_{(fg)^n y}^{-1}[B_n((fg)^n y,\rho_{\epsilon,g^{-1}},g^{-1})\cap B((fg)^n y,\epsilon e^{-2n\gamma})]\subset\tilde{B}(\epsilon l e^{-2(n-k)\gamma}).$$
Using $n-k$ instead of k, and noticing $(\widetilde{fg})^n\tilde{g}_y^{-(n-k)}=(\widetilde{fg})^k\tilde{f}_y^{n-k}$, we have
$$\tilde{f}_y^{n-k}E\subset\tilde{B}(\epsilon l e^{-2k\gamma}),\quad (\widetilde{fg})^k\tilde{f}_y^{n-k}E\subset\tilde{B}(\epsilon l e^{-2k\gamma}),\quad\forall\ k=0,1,\cdots,n.\tag{$*$}$$

Now we claim that $\forall\ k = 0, 1, \cdots, n$,

$$(\widetilde{fg})^i \tilde{f}_y^{n-k} E \subset \tilde{B}(\epsilon l e^{-2\max\{k-i,i\}\gamma}), \qquad \forall\ i = 0, 1, \cdots, k.$$

For $k = 0$, the claim is true because by $(*)$ we have already had $\tilde{f}_y^n E \subset B(\epsilon l)$. We suppose the claim is true for $k-1$, i.e.

$$(\widetilde{fg})^i \tilde{f}_y^{n-k+1} E \subset \tilde{B}(\epsilon l e^{-2\max\{k-i-1,i\}\gamma}), \qquad \forall\ i = 0, 1, \cdots, k-1.$$

Thus

$$(\widetilde{fg})^i \tilde{f}_y^{n-k} E \subset \tilde{f}^{-1} \tilde{B}(\epsilon l e^{-2\max\{k-i-1,i\}\gamma}), \qquad \forall\ i = 0, 1, \cdots, k-1. \tag{$**$}$$

Since $0 < \epsilon < l^{-2} e^{-\lambda - 2\gamma}$, $\epsilon l e^{-2\max\{k-i-1,i\}\gamma} e^{\lambda} < l^{-1} e^{-2\max\{k-i,i\}\gamma}$. Also,

$$l((fg)^i f^{n-k} y) = l(f^{-(k-i)} g^i (f^n y)) \le l(f^n y) e^{(k-i)\gamma} e^{i\gamma} \le l e^{k\gamma} \le l e^{2\max\{k-i,i\}\gamma}.$$

Therefore $\epsilon l e^{-2\max\{k-i-1,i\}\gamma} e^{\lambda} < l((fg)^i f^{n-k} y)^{-1}$. Now we can use Corollary 4.2 on the right hand side in $(**)$ and obtain that $\forall\ i = 0, 1, \cdots, k-1$,

$$(\widetilde{fg})^i \tilde{f}_y^{n-k} E \subset \tilde{B}(\epsilon l e^{-2\max\{k-i-1,i\}\gamma} e^{\lambda}) \subset \tilde{B}(l((fg)^i f^{n-k} y)^{-1}). \tag{$***$}$$

By Proposition 4.1, $l(f^{\pm 1} x) \le l(x) e^{\gamma}$. Hence $y \in \Gamma_l \cap f^{-n}\Gamma_l$ implies that $f^{n-k} y \in \Gamma_{le^{k\gamma}} \cap f^{-k}\Gamma_l \subset \Gamma_{l'} \cap f^{-k}\Gamma_{l'}$, where $l' = l e^{k\gamma}$. Also, $\epsilon l e^{-2k\gamma} < (l')^{-1}$. Thus by $(*)$ and $(***)$ we can use Lemma 7.1.ii) on the set $\tilde{f}_y^{n-k} E$ for diffeomorphism fg to obtain that

$$(\widetilde{fg})^i \tilde{f}_y^{n-k} E \subset \tilde{B}(\epsilon l e^{-2\max\{k-i,i\}\gamma}), \qquad \forall\ i = 0, 1, \cdots, k,$$

i.e. the claim is true for k.

By induction the claim is true for $k = n$. So

$$(\widetilde{fg})^i E \subset \tilde{B}(\epsilon l e^{-2\max\{n-i,i\}\gamma}) \subset \tilde{B}(\epsilon l), \qquad \forall\ i = 0, 1, \cdots, n.$$

Thus,

$$(fg)^i \Phi_y E = \Phi_{(fg)^i y} (\widetilde{fg})^i_y E \subset \Phi_{(fg)^i y} \tilde{B}(\epsilon l) \subset B((fg)^i y, \epsilon l K), \qquad \forall\ i = 0, 1, \cdots, n,$$

i.e.

$$\Delta = \Phi_y E \subset (fg)^{-i} B((fg)^i y, \epsilon l K), \qquad \forall\ i = 0, 1, \cdots, n.$$

The result follows from the definition of $B_n(y, \epsilon l K, fg)$. ∎

The Proof of Theorem B (First Part).

Because the entropy map, which is defined on the set of invariant measures and has values in $[0, +\infty]$, is affine and any $\mu \in \mathcal{M}(M, f, g)$ has $(f, g)-$ergodic decomposition, we only need prove the theorem if μ is an $(f, g)-$ergodic measure.

Take $\gamma > 0$ small sufficiently.

Take $\delta \in (0, 1)$.

Let $\{\Phi_x : x \in \Gamma\}$ be a system of $(\gamma, l)-$charts for both f and g. Let $\Gamma_l = \{x \in \Gamma : l(x) \leq l\}$. Fix an $l > 1$ such that $\mu\Gamma_l > 1 - \frac{\delta}{5}$. We define two families of functions $\{\rho_{\epsilon,f}\}$ and $\{\rho_{\epsilon,g^{-1}}\}$ as above corresponding to f and g^{-1} respectively.

Let

$$A^f_{n,\epsilon,\gamma} = \{x \in \Gamma : \ \mu B_k(x, \rho_{\epsilon,f}, f) \geq \exp -k(h_\mu(f) + \gamma), \quad \forall k \geq n\}.$$

Since $\rho_{\epsilon,f}$ is decreasing as $\epsilon \to 0$, by Proposition 6.4,

$$h_\mu(f) = h_\mu(x, f) \geq \limsup_{n\to\infty} -\frac{1}{n} \log \mu B_n(x, \rho_{\epsilon,f}, f), \quad \mu - a.e. \ x \in \Gamma.$$

So $\forall \ \epsilon > 0$, $\mu A^f_{n,\epsilon,\gamma} \to 1$, as $n \to \infty$. Then $\exists \ n_f(\epsilon) > 0$, such that $\forall \ n > n_f(\epsilon)$,

$$\mu A^f_{n,\epsilon,\gamma} \geq 1 - \frac{\delta}{5}.$$

By the definition of $A^f_{n,\epsilon,\gamma}$, there are at most $\exp n(h_\mu(f) + \gamma)$ disjoint $(n, \rho_{\epsilon,f}, f)-$balls centered at points in $A^f_{n,\epsilon,\gamma}$. So the same number of $(n, 2\rho_{\epsilon,f}, f)-$balls centered at points in $A^f_{n,\epsilon,\gamma}$ can cover $A^f_{n,\epsilon,\gamma}$. Suppose $\{B_n(x, 2\rho_{\epsilon,f}, f) : x \in S_f\}$ is a set of such balls. Then we have that

$$\bigcup_{x\in S_f} B_n(x, 2\rho_{\epsilon,f}, f) \supset A^f_{n,\epsilon,\gamma},$$

$$|S_f| \leq \exp n(h_\mu(f) + \gamma).$$

Similarly, for diffeomorphism g^{-1}, $\forall \ \epsilon > 0$, $\exists n_g(\epsilon) > 0$, such that $\forall \ n > n_g(\epsilon)$, we have sets $A^g_{n,\epsilon,\gamma}$ and S_g satisfying the following.

$$\mu A^g_{n,\epsilon,\gamma} \geq 1 - \frac{\delta}{5}.$$

$$\bigcup_{x\in S_g} B_n(x, 2\rho_{\epsilon,g^{-1}}, g^{-1}) \supset A^g_{n,\epsilon,\gamma},$$

$$|S_g| \leq \exp n(h_\mu(g) + \gamma).$$

We denote by $N(\alpha)$ the minimal number of balls of radius α covering M. Since $\dim M = m$, there exists a constant $C > 0$ such that $N(\alpha) < C\alpha^{-m}$, $\forall \ \alpha > 0$. Let S_0 be a set such that

$$\bigcup_{x\in S_0} B(x, 2\epsilon e^{-2n\gamma}) \supset M,$$

$$|S_0| = N(2\epsilon e^{-2n\gamma}) < C \cdot (2\epsilon e^{-2n\gamma})^{-m}.$$

Now we take $0 < \epsilon < \frac{1}{4} l^{-2} e^{-\lambda-2\gamma}$. For each $n > \max\{n_f(\epsilon), n_g(\epsilon)\}$, let

$$A_n = A^f_{n,\epsilon,\gamma} \cap A^g_{n,\epsilon,\gamma} \cap \Gamma_l \cap f^{-n}\Gamma_l \cap (fg)^{-n}\Gamma_l.$$

Clearly, $\mu A_n \geq 1 - \delta$. For any $x_f \in S_f$, $x_g \in S_g$, $x', x'' \in S$, if the intersection

$$A_n \cap B_n(x_f, 2\rho_{\epsilon,f}, f) \cap B(x', 2\epsilon e^{-2n\gamma}) \bigcap (fg)^{-n} \Big[B_n(x_g, 2\rho_{\epsilon,g^{-1}}, g^{-1}) \cap B(x'', 2\epsilon e^{-2n\gamma}) \Big]$$

is not empty, then for any y in it, the intersection is contained in the set

$$B_n(y, 4\rho_{\epsilon,f}, f) \cap B(y, 4\epsilon e^{-2n\gamma}) \bigcap (fg)^{-n}\Big[B_n((fg)^n y, 4\rho_{\epsilon,g^{-1}}, g^{-1}) \cap B((fg)^n y, 4\epsilon e^{-2n\gamma})\Big].$$

Notice $y \in A_n \subset \Gamma_l \cap f^{-n}\Gamma_l \cap (fg)^{-n}\Gamma_l$, and $0 < 4\epsilon < l^{-2}e^{-\lambda-2\gamma}$, the result of Lemma 7.3 still holds if we use 4ϵ instead of ϵ. So the set is contained in $B_n(y, 4\epsilon lK, fg)$.

There are at most

$$\begin{aligned}|S_f| \cdot |S_g| \cdot |S_0|^2 &\leq \exp n(h_\mu(f) + \gamma) \cdot \exp n(h_\mu(g) + \gamma) \cdot C^2 \cdot (2\epsilon e^{-2n\gamma})^{-2m} \\ &= C^2 \cdot (2\epsilon)^{-2m} \cdot \exp n[h_\mu(f) + h_\mu(g) + (4m+2)\gamma]\end{aligned}$$

different such intersections. Each one is contained in an $(n, 4\epsilon lK, fg)-$ball. Since these intersections cover A_n, and $\mu A_n > 1 - \delta$, we have

$$N_n(4\epsilon lK, \delta, fg) \leq C^2 \cdot (2\epsilon)^{-2m} \cdot \exp n[h_\mu(f) + h_\mu(g) + (4m+2)\gamma].$$

Thus,

$$h_\mu(fg) = \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log N_n(4\epsilon lK, \delta, fg) \leq h_\mu(f) + h_\mu(g) + (4m+2)\gamma.$$

Since γ is arbitrary, we get

$$h_\mu(fg) \leq h_\mu(f) + h_\mu(g).$$

∎

§8. A Partition Subordinating to W^u-foliations

In this section we assume that $E^u(x,f) = E^u(x,g),\ \mu - a.e.$ By Proposition 5.4, we have $w^u(x,f) = w^u(x,g),\ \mu - a.e.$ Therefore it can be written as $w^u(x)$. We will construct a measurable partition η subordinating to w^u and increasing under the action of diffeomorphisms f and g, so that we can compute entropies of the diffeomorphisms and get the proof of the equality part in Theorem B.

In our discussion we also assume that μ is an $(f,g)-$ergodic measure on M.

A measurable partition ξ of M is a partition of M such that, up to a set of measure zero, the quotient space M/ξ is separated by a countable number of measurable sets(see[Ro]).

A measurable partition ξ of M is said to be subordinate to the w^u-foliation if for $\mu - a.e.x$, $\xi(x) \subset w^u(x)$ and $\xi(x)$ contains a neighborhood of x open in the submanifold topology of $w^u(x)$(see [LY]).

For two partitions ξ_1 and ξ_2, we say ξ_1 refines ξ_2, denoted by $\xi_1 \geq \xi_2$, if $\xi_1(x) \subset \xi_2(x), \mu - a.e.$ We say that a partition ξ is $f-$increasing if $f\xi \leq \xi$, $g-$ increasing is defined analogously. ξ is said to be $(f,g)-$increasing, if ξ is both $f-$ and $g-$increasing.

Let $\mathcal{B}^u$ be the biggest sub$-\sigma-$algebra whose elements are unions of entire w^u-manifold.

Proposition 8.1. *There is a measurable partition η on M with the following properties.*
i) η is subordinate to w^u-foliation.
ii) η is $(f,g)-$increasing, i.e. $f\eta \leq \eta$ and $g\eta \leq \eta$.

iii) Both $\bigvee_{n=0}^{\infty} f^{-n}\eta$ *and* $\bigvee_{k=0}^{\infty} g^{-k}\eta$ *are the partition into points (mod 0).*

iv) The biggest $\sigma-$*algebra contained in* $\bigcap_{n=0}^{\infty}\bigcap_{k=0}^{\infty} f^{-n}g^{-k}\eta$ *is* $\mathcal{B}^u$.

To prove the proposition we introduce some lemmas.

Lemma 8.2. *Let* $\rho_0 > 0$, $0 < a < 1$ *and* ν *be a finite non-negative Borel measure on* $[0, \rho_0]$. *Then the Lebesgue measure of the set*

$$L_a = \{\rho : \ 0 \le \rho \le \rho_0, \ \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \nu([\rho - a^{n+k}, \rho + a^{n+k}]) < \infty\}$$

is equal to ρ_0.

Proof. The idea is similar to the proof of Proposition 3.2 in [LS]. The modification is replacing set $N_{a,n}$ in the proof by set

$$N_{a,n,k} = \{\rho : \ 0 \le \rho \le \rho_0, \ \nu([\rho - a^{n+k}, \rho + a^{n+k}]) > \frac{\nu([0,\rho])}{n^2k^2}\}. \qquad \blacksquare$$

Lemma 8.3. *There is a constant* $b > 0$, *such that* $\forall x \in \Gamma$, *for Lebesgue almost every choice of* ρ, $0 < \rho < l(x)^{-1}$, $\mu - a.e. y \in M$, *the inequality*

$$d(f^{-n}g^{-k}y, \partial B(x,\rho))e^{(n+k)(\lambda_+(x)-2\gamma)} < b^{-1}$$

holds at most for finite number of pairs (n,k), *where* $\lambda_+(x) = \min\{\lambda_+(x,f), \lambda_+(x,g)\}$.

Proof. Take $b > 0$ such that $d(z, \partial B(x,\rho)) \le \tau$ implies $|d(x,z) - \rho| \le b\tau$ whenever $0 < \tau < \rho \le l(x)^{-1}$.

Define a non-negative Borel measure ν on $\mathbb{R}$ by $\nu(A) = \mu\{y \in M : \ d(x,y) \in A\}$ for any Borel set $A \subset \mathbb{R}$. Thus, by Lemma 8.2, we get, applied $a = e^{-(\lambda_+(x)-2\gamma)}$, that

$$P = \{\rho : \ 0 \le \rho \le l(x)^{-1}, \ \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \mu\{y \in M : \ |d(x,y) - \rho| < e^{-(n+k)(\lambda_+(x)-2\gamma)}\} < \infty\}$$

has Lebesgue measure $l(x)^{-1}$. Since μ is $f-$ and $g-$invariant,

$$P = \{\rho : \ 0 \le \rho \le l(x)^{-1}, \ \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \mu\{y \in M : \ |d(x, f^{-n}g^{-k}y) - \rho| < e^{-(n+k)(\lambda_+(x)-2\gamma)}\} < \infty\}.$$

From the choice of b, $\forall \rho \in P$,

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \mu\{y \in M : \ d(f^{-n}g^{-k}y, \partial B(x,\rho)) < \frac{1}{b}e^{-(n+k)(\lambda_+(x)-2\gamma)}\} < \infty.$$

By Borel-Cantelli Lemma, except for finite number of pairs (n,k),

$$\mu\{y \in M : \ d(f^{-n}g^{-k}y, \partial B(x,\rho))e^{(n+k)(\lambda_+(x)-2\gamma)} < \frac{1}{b}\} = 0.$$

This finishes the proof of the lemma. ∎

Define a metric $d_w(\cdot,\cdot)$ on M by

$$d_w(y,z) = \begin{cases} d_{w(x)}(y,z), & \text{if } y,z \in w^u(x) \text{ for some } x \in M; \\ \infty, & \text{otherwise,} \end{cases}$$

where $d_{w(x)}(\cdot,\cdot)$ is a metric on $w^u(x)$ induced by the Riemannian structure on $w^u(x)$. Clearly, $d_w(\cdot,\cdot)$ is independent of the choice of $x \in M$.

Lemma 8.4. *For $0 < \alpha < 1$, $z \in \Gamma$, if $y \in w^u_\alpha(z)$, then $\forall n,k \geq 0$,*

$$d_w(f^{-n}g^{-k}y, f^{-n}g^{-k}z) \leq 2Kl(z)d_w(y,z)e^{-(n+k)(\lambda_+(x)-2\gamma)}.$$

Proof. Take a system of $(\gamma,l)-$charts $\{\Phi_z : z \in \Gamma\}$ for both f and g. Let $v = \Phi_z^{-1}y$, then $v \in W^u_\alpha(z)$. By Lemma 5.3,

$$|(\tilde{f}^{-n}\tilde{g}^{-k})_z v| \leq |v|e^{-n\lambda_+(z,f)-k\lambda_+(z,g)-2(n+k)\gamma} \leq |v|e^{-(n+k)(\lambda(z)-2\gamma)}.$$

By Proposition 4.1.vi) and Lemma 5.1, $d_w(f^{-n}g^{-k}y, f^{-n}g^{-k}z) \leq 2K|(\tilde{f}^{-n}\tilde{g}^{-k})_z v|$ and $|v| \leq l(z)d(y,z) \leq l(z)d_w(y,z)$. Hence the result follows. ∎

The Proof of Proposition 8.1.

Take $l > 0$ with $\mu\Gamma_l > 0$.

Fix $0 < \alpha < 1$. Take $x \in \Gamma_l$ such that $\forall \rho > 0$, $\mu(B(x,\rho) \cap \Gamma_l) > 0$.

Let

$$S(x,\rho) = \bigcup_{y \in \Gamma_l \cap B(x,\rho)} w^u_\alpha(y) \cap B(x,\rho).$$

Thus $\forall\ 0 < \rho < \frac{\alpha}{4}l^{-1}$, if two points $z_1, z_2 \in S(x,\rho)$ are not in the same local leaf $w^u_\alpha(y) \cap B(x,\rho)$ for some $y \in \Gamma_l \cap B(x,\rho)$, then $d_w(z_1,z_2) > 2\rho$.

For any $0 < \rho < \frac{\alpha}{4}l^{-1}$ we construct a partition ξ_ρ of M defined by all the sets

$$\xi_\rho(y) = \begin{cases} w^u_\alpha(y) \cap S(x,\rho), & \text{if } y \in S(x,\rho); \\ M \backslash S(x,\rho), & \text{otherwise,} \end{cases}$$

and then put

$$\eta_\rho = \bigvee_{n=0}^{\infty} \bigvee_{k=0}^{\infty} f^n g^k \xi_\rho.$$

Since $\mu(\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{\infty} f^n g^k S(x,\rho)) = 1$, it follows that $\mu - a.e. z \in \Gamma$, $\eta_\rho(z) \subset w^u(z)$. It is also clear that η_ρ satisfies the properties ii)$-$iv) in the proposition. To complete the proof we have to choose a $\rho > 0$ such that $\mu - a.e. z, \eta_\rho(z)$ contains an open neighborhood of z in the submanifold topology of $w^u(z)$.

Let

$$\beta_\rho = l(z)^{-1} \cdot \inf_{n \geq 0}\{\alpha, \frac{1}{4}K^{-1}d(f^{-n}g^{-k}z, \partial B(x,\rho))e^{(n+k)(\lambda_+(z)-2\gamma)}, K^{-1}\rho\}.$$

By Lemma 8.3, there is a $\rho > 0$ such that $\beta_\rho(z) > 0$, $\mu - a.e.\ z \in M$.

Now we only need prove that $\forall z \in \Gamma$, if $y \in w^u(z), d_w(y,z) < \beta_\rho(z)$, then $y \in \eta_\rho(z)$. In this circumstances, $y \in w^u_\alpha(z)$ and by Lemma 8.4, $\forall\ n, k \geq 0$,

$$d_w(f^{-n}g^{-k}y, f^{-n}g^{-k}z) \leq 2Kl(z)e^{-(n+k)(\lambda_+(z)-2\gamma)} \cdot \beta_\rho(z).$$

We have following cases to consider.

i) Both $f^{-n}g^{-k}y$ and $f^{-n}g^{-k}z$ belong to $S(x,r)$. By the choice of $\beta_\rho(z)$,

$$d_w(f^{-n}g^{-k}y, f^{-n}g^{-k}z) \leq 2\rho.$$

So the two points in the same local leaf of $S(x,\rho)$. Therefore $\xi_\rho(f^{-n}g^{-k}y) = \xi_\rho(f^{-n}g^{-k}z)$.

ii) Neither $f^{-n}g^{-k}y$ nor $f^{-n}g^{-k}z$ belongs to $S(x,r)$. By the construction of ξ_ρ, we have $\xi_\rho(f^{-n}g^{-k}y) = \xi_\rho(f^{-n}g^{-k}y)$.

iii) One of $f^{-n}g^{-k}y$ and $f^{-n}g^{-k}z$ belongs to $S(x,r)$ but the other does not. By the choice of $\beta_\rho(z)$,

$$d_w(f^{-n}g^{-k}y, f^{-n}g^{-k}z) \leq \frac{1}{2}d(f^{-n}g^{-k}z, \partial B(x,\rho)).$$

It is impossible.

Hence $\forall\ n, k \geq 0$, we always have $\xi_\rho(f^{-n}g^{-k}y) = \xi_\rho(f^{-n}g^{-k}z)$. So $y \in \eta_\rho(z)$. ∎

We fix $\rho > 0$ such that $\eta = \bigvee_{n=0}^{\infty}\bigvee_{k=0}^{\infty} f^n g^k \xi_\rho$ is the measurable partition of M satisfying Proposition 8.1.

Lemma 8.5. *Let η be a partition constructed as above. Then*

$$h_\mu(f,\eta) = h_\mu(f), \quad h_\mu(g,\eta) = h_\mu(g).$$

Proof. We only prove the first equality.

For any $f-$invariant measure ν, $h_\nu(f,\eta) = H_\nu(\eta|f\eta) = \int_M -\log\nu(\eta(x)|f\eta(x))d\nu(x)$, where $\nu(\cdot|\eta(x))$ is the system of conditional measures with respect to the $\sigma-$algebra generated by partition η. By $(\Delta\Delta)$ in §1 we have $h_\mu(f,\eta) = \int_{\mathcal{E}(M,f)} h_{\mu_e}(f,\eta)d\pi(\mu_e)$. Similarly, $h_\nu(f) = \int_M h_\nu(x,f)d\nu(x)$ implies $h_\mu(f) = \int_{\mathcal{E}(M,f)} h_{\mu_e}(f)d\pi(\mu_e)$. Therefore we only need show that $\pi - a.e.\ \mu_e$, $h_{\mu_e}(f,\eta) = h_{\mu_e}(f)$.

For $\pi - a.e.\ \mu_e$, $\mu_e(\bigcup_{n=0}^{\infty}\bigcup_{k=0}^{\infty} f^n g^k S(x,\rho)) = 1$. Since $\mu_e \in \mathcal{E}(M,f)$, we can take $k = k(\mu_e)$ such that $\mu_e(\bigcup_{n=0}^{\infty} f^n g^k S(x,\rho)) = 1$. Denote $\eta_e = \bigvee_{n=0}^{\infty} f^n g^k \xi_\rho$, then η_e is a partition constructed as same as in the proof of Lemma 3.1.1 in [LY]. So we have $h_{\mu_e}(f,\eta_e) = h_{\mu_e}(f)$ (see [LY], Corollary 5.3).

Now we need prove that $h_{\mu_e}(f,\eta_e) = h_{\mu_e}(f,\eta)$. The argument is similar to the prove of Lemma 3.1.2 in [LY]. Notice $\eta_e \leq \eta$ and $f\eta_e \leq \eta_e$, we have

$$\begin{aligned} h_{\mu_e}(f,\eta) = h_{\mu_e}(f,\eta_e \vee \eta) = h_{\mu_e}(f,\eta_e \vee f^n\eta) &= H_{\mu_e}(\eta_e \vee f^n\eta | f\eta_e \vee f^{n+1}\eta) \\ &= H_{\mu_e}(\eta_e | f\eta_e \vee f^{n+1}\eta) + H_{\mu_e}(\eta_e | f^{-n}\eta_e \vee f\eta). \end{aligned}$$

As $n \to \infty$, the first term increases monotonously and tends to $H_{\mu_e}(\eta_e|f\eta_e) = h_{\mu_e}(f, \eta_e)$, while the second term decreases and goes to 0. Since the formula is true for any $n > 0$, this finishes the proof. ∎

§9. The Condition for Equality

We will prove the rest part of Theorem B in the section. First we consider a special case, for $\mu - a.e, x \in M$, $E^u(x, f) = E^u(x, g)$.

Proposition 9.1. *If $E^u(x, f) = E^u(x, g), \mu - a.e.$ then $h_\mu(fg) = h_\mu(f) + h_\mu(g)$.*

Proof. Take partition η as in the proof of Proposition 8.1. Since $f\eta \le \eta$ and $g\eta \le \eta$, by Lemma 8.5, we have

$$h_\mu(fg) \ge h_\mu(fg, \eta) = H_\mu(\eta|fg\eta) = H_\mu(\eta \vee g\eta|fg\eta)$$
$$= H_\mu(g\eta|fg\eta) + H_\mu(\eta|g\eta \vee fg\eta) = H_\mu(\eta|f\eta) + H_\mu(\eta|g\eta) = h_\mu(f) + h_\mu(g).$$

Then the result follows from the first part of Theorem B. ∎

Now we consider the general case, $E^u(x, f) \cap E^s(x, g) = \{0\} = E^s(x, f) \cap E^u(x, g), \mu - a.e.$

Lemma 9.2. *Suppose $E^u(x, f) \cap E^s(x, g) = \{0\}$ and $E^s(x, f) \cap E^u(x, g) = \{0\}, \mu - a.e.$ Then there exists an $N > 0$, such that $\forall\ n > N$,*

$$h_\mu(f^n g) \ge h_\mu(f^n), \qquad h_\mu(fg^n) \ge h_\mu(g^n).$$

Proof. Take $N_f > 0$, such that

$$N_f \cdot \Delta\lambda(x, f) > \lambda_1(x, g),$$

where $\Delta\lambda(x, f) = \min\{\lambda_i(x, f) - \lambda_{i+1}(x, f) :\ i = 1, \cdots, s(x, f) - 1\}$ is as in §4.

Let $u(x, f) = \min\{i : \lambda_i(x, f) \ge 0\}$, in other words, $u(x, f)$ is defined such that $\lambda_{u(x,f)}(x, f)$ is the smallest nonnegative exponent of f. $u(x, g)$, $u(x, f^n g)$ are understood in similar way.

For any $n \ge N_f$, if we denote the Lyapunov exponents of $f^n g$ by $\lambda_1(x, f^n g) > \cdots > \lambda_{r(x,f^n g)}(x, f^n g)$, then by Theorem A and the supposition of the lemma, $\forall\ 1 \le p \le u(x, f^n g)$, $\exists\ 1 \le i \le u(x, f)$, $1 \le j \le u(x, g)$ with

$$\lambda_p(x, f^n g) = n\lambda_i(x, f) + \lambda_j(x, g).$$

By the choice of N_f we know that $n\lambda_{i_1}(x, f) + \lambda_{j_1}(x, g) > n\lambda_{i_2}(x, f) + \lambda_{j_2}(x, g)$ if and only if $i_1 > i_2$, or $i_1 = i_2$ and $j_1 > j_2$. Thus the decomposition $E^u(x, f^n g) = \bigoplus_{u=1}^{u(x,f^n g)} E_p(x, f^n g)$ of unstable part in tangent space can be written as

$$E^u(x, f^n g) = \bigoplus_{i=1}^{u(x,f)} \bigoplus_{j=1}^{u(x,g)} E_{ij}(x),$$

where $E_{ij}(x) = E_p(x, f^n g)$, if $n\lambda_i(x, f) + \lambda_j(x, g) = \lambda_p(x, f^n g)$ for some $1 \le p \le u(x, f^n g)$, and otherwise $E_{ij}(x) = \{0\}$. The Ledrappier-Young's formula relating entropy, exponents and dimensions is

$$h_\mu(x, f) = \sum_{i=1}^{u(x,f)} \lambda_i(x, f)\gamma_i(x, f),$$

where $\gamma_i(x, f)$ denotes a notion of fractional dimension defined as follows.

Let $w^{(i)}(x, f) = \{y \in M : \limsup_{n\to\infty} \frac{1}{n} \log d(f^{-n}x, f^{-n}y) \le -\lambda_i(x, f)\}$ for each i with $\lambda_i(x, f) > 0$, which is a C^2 immersed submanifold of M with dimension $\sum_{s\le i} \dim E_s(x, f)$, and $\eta^{(i)}$ be a partition subordinating to $\{w^{(i)}(x, f)\}$. Denote by $B^{(i)}(x, \epsilon)$ the ball in $w^{(i)}(x, f)$ centered at x of radius ϵ in the distance induced by the Riemannian structure on $w^{(i)}(x, f)$. For each $i = 1, \cdots, u(x, f)$ with $\lambda_i(x, f) > 0$, define

$$\delta_i(x, f) = \lim_{\epsilon\to 0} \frac{\log \mu(B^{(i)}(x, \epsilon)|\eta^{(i)}(x))}{\log \epsilon},$$

where the limits in the right hand side exist $\mu-$ a.e. $x \in M$ and are independent of the choice of η (See [LY], §7). And then let

$$\gamma_i(x, f) = \delta_i(x, f) - \delta_{i-1}(x, f), \qquad i = 1, \cdots, u(x, f),$$

where we regard $\delta_0(x, f) = 0$, and $\delta_{u(x,f)}(x, f)$ as any fixed constant between $\delta_{u(x,f)-1}(x, f)$ and $\dim M$ if $\lambda_{u(x,f)}(x, f) = 0$. $\gamma_i(x, g)$ and $\gamma_i(x, f^n g)$ are defined similarly.

For any $1 \le i \le u(x, f)$, let p_i be the smallest number such that $\lambda_{p_i}(x, f^n g) \ge n\lambda_i(x, f)$, and let $p_0 = 0$. It means that if $p_i \le p < p_{i+1}$, then $\lambda_p(x, f^n g) = n\lambda_i(x, f) + \lambda_j(x, g)$ for some $1 \le j \le u(x, g)$. Thus $w^{(i)}(x, f) = w^{(p_i)}(x, f^n g)$ and $\delta_i(x, f) = \delta_{p_i}(x, f^n g)$. So

$$\gamma_i(x, f) = \delta_{p_i}(x, f^n g) - \delta_{p_{i-1}}(x, f^n g) = \sum_{p=p_{i-1}+1}^{p_i} \gamma_p(x, f^n g)$$

except for $i = u(x, f)$ as $\lambda_{u(x,f)}(x, f) = 0$. Put $\gamma_{ij}(x) = \gamma_p(x, f^n g)$ if $E_{ij}(x) = E_p(x, f^n g)$, and $\gamma_{ij}(x) = 0$ if $E_{ij}(x) = \{0\}$. Above formula shows that $\gamma_i(x, f) = \sum_{j=1}^{u(x,g)} \gamma_{ij}(x)$. Now the Ledrappier-Young's formula for diffeomorphism $f^n g$ can be expressed as

$$h_\mu(x, f^n g) = \sum_{p=1}^{u(x,f^n g)} \lambda_p(x, f^n g)\gamma_p(x, f^n g) = \sum_{i=1}^{u(x,f)} \sum_{j=1}^{u(x,g)} (n\lambda_i(x, f) + \lambda_j(x, g))\gamma_{ij}(x).$$

Therefore

$$h_\mu(x, f^n g) \ge \sum_{i=1}^{u(x,f)} \sum_{j=1}^{u(x,g)} n\lambda_i(x, f)\gamma_{ij}(x) = \sum_{i=1}^{u(x,f)} n\lambda_i(x, f)\gamma_i(x) = nh_\mu(f).$$

Similarly we have $N_g > 0$, such that $\forall\, n > N_g$, $h_\mu(fg^n) \geq h_\mu(g^n)$. Then $N = \max\{N_f, N_g\}$ is a required number. ∎

The Proof of Theorem B (Second Part).

Notice for any $n > 0$, $E^u(x, f^n g) = E^u(x, fg^n)$, $\mu - a.e.$ because of the supposition of the theorem. By Proposition 9.1 and Lemma 9.2, if n is large sufficiently, then

$$\begin{aligned} h_\mu(fg) &= \frac{1}{n+1} h_\mu(f^n g \cdot fg^n) = \frac{1}{n+1}[h_\mu(f^n g) + h_\mu(fg^n)] \\ &\geq \frac{1}{n+1}[h_\mu(f^n) + h_\mu(g^n)] = \frac{n}{n+1}[h_\mu(f) + h_\mu(g)]. \end{aligned}$$

Since n is arbitrary, $h_\mu(fg) = h_\mu(f) + h_\mu(g)$. ∎

§10. Topological Entropy

By applying Theorem B, we can get similar result for topological entropies of commuting diffeomorphisms. We will use the relationship between topological entropies and measure-theoretic entropies, i.e. $h(f) = \max\{h_\nu(f) : \nu \in \mathcal{M}(M, f)\}$. Next proposition is in fact a generalization of Theorem C.

Proposition 10.1. *Suppose M is a compact C^∞ Riemannian manifold without boundary, f and g are commuting diffeomorphisms in $\mathrm{Diff}^2(M)$. If $\forall\, \alpha > 0$, $\exists\, 0 \leq \beta \leq \alpha$, such that the set*

$$\mathcal{V}_\beta = \{\nu \in \mathcal{M}(M, fg) : h_\nu(fg) \geq h(fg) - \beta\}$$

is compact in weak * topology, then

$$h(fg) \leq h(f) + h(g).$$

Proof. Since entropy map $\nu \to h_\nu(fg)$ is affine, $\mathcal{V}_\beta$ is a convex set. So $\mathcal{V}_\beta$ is a nonempty convex set which is compact in weak * topology. By Lemma 6.1, $f^*\mathcal{V}_\beta \subset \mathcal{V}_\beta$. So f^* has a fixed point in $\mathcal{V}_\beta$, i.e. $\mathcal{V}_\beta \cap \mathcal{M}(M, f) \neq \emptyset$.

Take $\mu \in \mathcal{V}_\beta \cap \mathcal{M}(M, f)$, then $\mu \in \mathcal{M}(M, f, fg)$. By Corollary 1.5, $\mu \in \mathcal{M}(M, f, g)$. Hence $h_\mu(fg) \leq h_\mu(f) + h_\mu(g) \leq h(f) + h(g)$. Also, $\mu \in \mathcal{V}_\beta$, so $h(fg) \leq h_\mu(fg) + \beta \leq h(f) + h(g) + \beta$. Since $0 \leq \beta \leq \alpha$ and α is arbitrary, our result follows. ∎

Now we consider two important cases, i.e. the entropy map of fg is upper semi-continuous and the set of measures with maximal entropy of fg is a finite dimensional simplex.

Corollary 10.2. *Suppose the entropy map of fg is upper semi-continuous, i.e. $\forall \nu_0 \in \mathcal{M}(M, fg)$ and $\beta > 0$, there exists a neighborhood $\mathcal{U}$ of ν_0 in $\mathcal{M}(M, fg)$ such that $\nu \in \mathcal{U}$ implies $h_\nu(fg) \leq h_{\nu_0}(fg) + \beta$. Then $h(fg) \leq h(f) + h(g)$.*

Proof. Since entropy map $\nu \to h_\nu(fg)$ is upper semi-continuous, $\forall\, \alpha \geq 0$, $\mathcal{V}_\alpha$, the preimage of $[h(fg) - \alpha, h(fg)]$ under the entropy map of fg, is compact. ∎

Corollary 10.3. *If the set of measures with maximal entropy of fg is a finite dimensional simplex, then $h(fg) \leq h(f) + h(g)$.*

Proof. This is because the set $\mathcal{V}_0 = \{\nu \in \mathcal{M}(M, fg) : h_\nu(fg) = h(fg)\}$ is compact. ∎

The Proof of Theorem C.

i) and ii) are from Corollary 10.2 plus [N] and [W] (Chapter 8) respectively, and iii) is Corollary 10.3. ∎

We end this paper by a counterexample of two commuting homeomorphisms f and g with zero entropies on a smooth manifold $M = S^1 \times S^2$ whose composition has positive entropy.

Example. Let $S^1 = \{\theta \in [0, 2\pi] : \{0\} = \{2\pi\}\ \}$. Take a homeomorphism $\alpha : S^1 \to S^1$, such that $\alpha(0) = 0$ and $\alpha(\theta) \leq \theta$, $\forall\ \theta \in (0, 2\pi)$. Hence α has unique fixed point $\theta = 0$.

Let $S^2 = \{(r, \tau) \in [0, 2] \times [0, 2\pi] : \{(r, 0)\} = \{(r, 2\pi)\}, \forall r \in [0, 2]; \{(r, \tau)\} = \{(r, 0)\}, \forall \tau \in [0, 2\pi], r = 0, 2\}$. Take $D = \{(r, \tau) \in S^2 :\ 0 \leq r \leq 1\}$. Define a homeomorphism $\psi : S^2 \to S^2$, such that $\psi(D) = D$, restricted on D, $\psi|_D$ has positive entropy, and $\psi|_{S^2\setminus D} = id|_{S^2\setminus D}$.

Define a continuous map $\tilde{\beta} : (0, 2\pi) \times [0, 2] \to [0, 2]$, such that $\forall \theta \in (0, 2\pi)$, $\tilde{\beta}(\theta, \cdot) = \tilde{\beta}_\theta(\cdot) : [0, 2] \to [0, 2]$ is a homeomorphism, and for any $r \in [0, 1]$, $\tilde{\beta}_\theta(r) \leq \theta r$, if $\theta \in [0, \pi]$; $\tilde{\beta}_\theta(r) \leq (2\pi - \theta)r$, if $\theta \in [\pi, 2\pi]$. Then we use $\tilde{\beta}_\theta$ define a family of homeomorphisms β_θ on S^2 by putting

$$\beta_\theta(r, \tau) = (\tilde{\beta}_\theta(r), \tau), \qquad \forall\ (r, \tau) \in S^2.$$

Now we take $M = S^1 \times S^2$. Define $f, g : M \to M$ by

$$f(\theta, v) = (\alpha(\theta), \beta_{\alpha(\theta)}\psi\beta_\theta^{-1}(v)\ ),$$

$$g(\theta, v) = (\alpha^{-1}(\theta), \beta_{\alpha^{-1}(\theta)}\beta_\theta^{-1}(v)\ ),$$

for any $\theta \in S^1\setminus\{0\}$, $v = (r, \tau) \in S^2$, and

$$f(0, v) = (0, v) = g(0, v), \qquad \forall v \in S^2.$$

Clearly, both f and g are homeomorphisms under a suitable choice of $\tilde{\beta}_\theta$.

For $\theta \in S^1\setminus\{0\}$, $v \in S^2$,

$$\begin{aligned} fg(\theta, v) &= f(\alpha^{-1}(\theta), \beta_{\alpha^{-1}(\theta)}\beta_\theta^{-1}(v)\ ) \\ &= (\alpha\alpha^{-1}(\theta), \beta_{\alpha\alpha^{-1}(\theta)}\psi\beta_{\alpha^{-1}(\theta)}^{-1}\beta_{\alpha^{-1}(\theta)}\beta_\theta^{-1}(v)\ ) = (\theta, \beta_\theta\psi\beta_\theta^{-1}(v)\ ), \\ gf(\theta, v) &= g(\alpha(\theta), \beta_{\alpha(\theta)}\psi\beta_\theta^{-1}(v)\ ) \\ &= (\alpha^{-1}\alpha(\theta), \beta_{\alpha^{-1}\alpha(\theta)}\beta_{\alpha(\theta)}^{-1}\beta_{\alpha(\theta)}\psi\beta_\theta^{-1}(v)\ ) = (\theta, \beta_\theta\psi\beta_\theta^{-1}(v)\ ), \end{aligned}$$

and

$$fg(0, v) = (0, v) = gf(0, v), \qquad \forall\ v \in S^2.$$

So $fg = gf$ holds on M.

The nonwandering set for f and g are $\Omega(f) = \{(0, v) : v \in S^2\} = \Omega(g)$, and $f|_{\Omega(f)} = id = g|_{\Omega(g)}$. So $h(f) = 0 = h(g)$. But for any $\theta \neq 0$, restrict to the set $\{(\theta, v) : v \in S^2\}$, fg is conjugate to ψ. Therefore $h(fg) \geq h(\psi) \geq 0$.

Since the support of any $f-$ or $g-$invariant measure μ must be contained in the set $\{(0, v) : v \in S^2\}$, $h_\mu(fg)$ should equal to zero if $\mu \in \mathcal{M}(M, f, g)$. It means that the example does not violate Theorem B, though f, g are not diffeomorphisms.

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