The dimensions of non-conformal repeller and average conformal repeller

Jungchao Ban

Department of mathematics, National Hualien University of Education

Hualien 97003, Taiwan, jcban@mail.nhlue.edu.tw

Yongluo Cao*

Department of mathematics, Suzhou University

Suzhou 215006, Jiangsu, P.R.China and

Institute of Mathematics, Fudan University

Shanghai, 200433, P.R. China

ylcao@suda.edu.cn, yongluocao@yahoo.com

Huyi Hu

Department of mathematics, Michigan state University

East Lansing, MI 48824, USA, hu@math.msu.edu

February 19, 2012

Abstract. In this paper, using thermodynamic formalism for sub-additive potential defined in [4], upper bounds of Hausdorff dimension and box dimension of non-conformal repellers are obtained as sub-additive Bowen equation. The map f is only needed C^1, without additional condition. We also proved that all the upper bounds of Hausdorff dimension in [1, 18, 8] are coincide. This unifies their results. Furthermore we define average conformal repeller and prove that the dimension of average conformal repeller equals to the unique root of sub-additive Bowen equation.

Key words and phrases Hausdorff dimension, Non-conformal repellers, Topological pressure.

---

[0]2000 *Mathematics Subject classification*: Primary 37D35; Secondary 37C45.

[0]* Corresponding author.

1 Introduction.

In the dimension theory of dynamical systems, it is very interesting topic to study the Hausdorff dimension of invariant sets of hyperbolic dynamics. Bowen [3] was the first to express the Hausdorff dimension of an invariant set as a solution of an equation involving topological pressure. Ruelle [13] refined Bowen's method and obtained the following result. Assume that f is a $C^{1+\gamma}$ conformal expanding map, Λ is an isolated compact invariant set and $f|_\Lambda$ is topologically mixing, then the Hausdorff dimension of Λ, $\dim_H \Lambda$ is given by the unique solution α of the equation

$$P(f|_\Lambda, -\alpha \log \|D_x f\|) = 0 \tag{1.1}$$

where $P(f|_\Lambda, \cdot)$ is the topological pressure functional. The smoothness $C^{1+\gamma}$ was recently relaxed to C^1 [9].

An estimate from above for the Hausdorff dimension of compact invariant sets for differentiable maps has been given by A.Douady and J.Oesterle [5], and by Ledreppier [11]. For non-conformal dynamical systems there exists only partial results. For example, the Hausdorff dimension of hyperbolic invariant sets was only computed in some special cases. Hu [10] gave an estimate of dimension of non-conformal repeller for C^2 map. Falconer [6, 7] computed the Hausdorff dimension of a class of non-conformal repellers. Related ideas were applied by Simon and Solomyak [16] to compute the Hausdorff dimension of a class of non-conformal horseshoes in R^3.

For C^1 non-conformal repellers, in [18], Zhang uses singular values of the derivative $D_x f^n$ for all $n \in Z^+$, to define a new equation which involves the limit of a sequence of topological pressure. Then he shows that the unique solution of the equation is an upper bounds of Hausdorff dimension of repeller. In [1], the same problem is considered. Barreira bases on the non-additive thermodynamic formalism which was introduced in [2] and singular value of the derivative $D_x f^n$ for all $n \in Z^+$, and gives an upper bounds of box dimension of repeller under the additional assumptions for which the map is $C^{1+\gamma}$ and γ-bunched. This automatically implies that for Hausdorff dimension. In [8], Falconer defines topological pressure of sub-additive potential under the condition $\|(D_x f)^{-1}\|^2 \|D_x f\| < 1$, which means that f is $1-$bunched. They also obtain an upper bounds of Hausdorff dimension of repeller. The questions are whether three bounds as above are the same and whether the upper bounds of box dimension holds true for C^1 non-conformal repeller?

In this paper, the first, using thermodynamic formalism for sub-additive potential defined in [4], we can obtain upper bounds of Hausdorff dimension and box dimension of non-conformal repellers. The map f is only needed C^1, without additional condition. In fact, we prove that the upper bounds of Hausdorff dimension of non-conformal repellers in [18] is the unique root of generalized Bowen equation which relates to sub-additive thermodynamic formalism. Furthermore, we proved all the upper bounds in [1, 18, 8] and ours are the same and we can prove that topological pressure in [4] is the same as in [1, 8] in which they need that f is $C^{1+\gamma}$ and γ-bunched condition. Our result also gives an affirmative answer to problem posed by K.Simon in [15] about an upper bound without assuming the 1-bunched property.

Then we introduce the notion of average conformal repeller. Using thermodynamic formalism for sub-additive potential, we prove that Hausdorff dimension and box dimension of average conformal repellers is the unique root of Bowen equation for sub-additive topological pressure. The map f is only needed C^1, without additional condition. Meanwhile, we introduce super-additive potential topological pressure and prove that for special potentials, sub-additive and super-additive topological pressures are same. In [2], Barreira introduces the concept of quasi-conformal repeller by using Markov construction and prove that its dimension is the unique root of the equation obtained by non-additive topological pressure. In [12] introduce the concept of weakly conformal repeller and obtain its dimension using Bowen equation. It is obvious that for C^1 map quasi-conformal and weakly conformal repeller are average conformal repellers, but reverse is not true. Therefore our result is a generalization of the results in [2, 12].

Next we recall some basic definitions and notations.

Let $f : X \to X$ be a continuous map. A set $E \subset X$ is called (n, ϵ) separated set with respect to f if $x, y \in E$ then $d_n(x, y) = \max_{0\le i\le n-1} d(f^i x, f^i y) > \epsilon$. For $x \in X$ and $r > 0$, define

$$B_n(x, r) = \{y \in X : f^i y \in B(f^i x, r), \text{ for all } i = 0, \cdots, n-1\}.$$

If ϕ is a real continuous function on X and $n \in Z^+$, let

$$S_n\phi(x) = \sum_{i=0}^{n-1} \phi(f^i(x)).$$

We define

$$P_n(\phi, \epsilon) = \sup\{\sum_{x\in E} \exp S_n\phi(x) : E \text{ is a } (n, \epsilon) - \text{separated subset of } X\}.$$

Then the topological pressure of ϕ is given by

$$P(f, \phi) = \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log P_n(\phi, \epsilon).$$

Next we give some properties of $P(f, \cdot) : C(M, R) \to R \cup \{\infty\}$.

Proposition 1.1. *Let $f : M \to M$ be a continuous transformation of a compact metrisable space M. If $\varphi_1, \varphi_2 \in C(X, R)$, then the followings are true:*

(1) $P(f, 0) = h_{top}(f)$.

(2) $|P(f, \varphi_1) - P(f, \varphi_2)| \le \|\varphi_1 - \varphi_2\|$.

(3) $\varphi_1 \le \varphi_2$ *implies that* $P(f, \varphi_1) \le P(f, \varphi_2)$.

Proof. See Walters book [17]. $\square$

Corollary 1. *Let $f : M \to M$ be a continuous transformation of a compact metrisable space M. If $\varphi \in C(M, R)$ and $\varphi < 0$ then function $P(\alpha) = P(f, \alpha\varphi)$ is continuous and strictly decreasing in α.*

Proof. Let $M = \max_{x \in M} \varphi(x)$ and $m = \min_{x \in M} \varphi(x)$. Then $\varphi \in C(M, R)$ and $\varphi < 0$ imply that $m \le M < 0$. If $\alpha_1 < \alpha_2$, then for all $n \in \mathbb{N}$, it has

$$(\alpha_2 - \alpha_1)nm \le S_n(\alpha_2\varphi)(x) - S_n(\alpha_1\varphi)(x) = (\alpha_2 - \alpha_1)S_n\varphi(x) \le (\alpha_2 - \alpha_1)nM.$$

Thus for $\forall \epsilon > 0$,

$$e^{(\alpha_2 - \alpha_1)nm} \times P_n(\alpha_1\varphi, \epsilon) \le P_n(\alpha_2\varphi, \epsilon) \le P_n(\alpha_1\varphi, \epsilon) \times e^{(\alpha_2 - \alpha_1)nM}.$$

It implies that

$$(\alpha_2 - \alpha_1)m + P(f, \alpha_1\varphi) \le P(f, \alpha_2\varphi) \le P(f, \alpha_1\varphi) + (\alpha_2 - \alpha_1)M.$$

Therefore $P(f, \alpha\varphi)$ is continuous and strictly monotone decreasing on α. $\square$

Another equivalent definition of topological pressure involves open covers.

Definition 1.1. *If $\varphi \in C(M, R)$, $n \ge 1$ and $\mathcal{U}$ is an open cover of M put*

$$p_n(f, \phi, \mathcal{U}) = \inf\{\sum_{\beta} \sup_{x \in B} e^{S_n\phi(x)} | \beta \text{ is a finite subcover of } \bigvee_{i=0}^{n-1} f^{-i}\mathcal{U}\}.$$

It is proved [17] that the limit

$$\lim_{n \to \infty} \frac{1}{n} \log p_n(f, \phi, \mathcal{U})$$

exists and equals to $\inf\limits_{n>0}\{\frac{1}{n} \log p_n(f, \varphi, \mathcal{U})\}$.

We have the following Lemma whose proof can be found in [17].

Lemma 1.1. *If $\phi \in C(M, R)$, $n \ge 1$ and $\mathcal{U}$ is an open cover of M, then*

$$\lim_{diam(\mathcal{U}) \to 0} \lim_{n \to \infty} \frac{1}{n} \log p_n(f, \phi, \mathcal{U}) = P(f, \phi).$$

A linear map $L : \mathbb{R}^n \to \mathbb{R}^n$ is said to be expanding if $\|Lv\| > \|v\|$ for all $v \in \mathbb{R}^n$ and $v \ne 0$. Given an expanding linear map $L : \mathbb{R}^m \to \mathbb{R}^m$, let $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_m \ge 0$ be the logarithms of the singular values of L, which are eigenvalues of $(L^*L)^{\frac{1}{2}}$, counted with their multiplicities, where $\lambda_m > 0$ because of the expansion. Following [5] we introduce the function

$$g^\alpha(L) = \sum_{i=m-[\alpha]+1}^{m} \lambda_i + (\alpha - [\alpha])\lambda_{m-[\alpha]},$$

for any $\alpha \in [0, m]$, where $[\alpha]$ is the largest integer $\leq \alpha$. $g^\alpha(L)$ is continuous and strictly increasing in α. $g^0(L) = 0$ and $g^m(L) = \sum_{i=1}^{m} \lambda_i = \log |Jac(L)|$, where $Jac(L)$ is the Jacobean of L. The map g^α has the following super-additive property. If $L : \mathbb{R}^m \to \mathbb{R}^m$ and $L' : \mathbb{R}^m \to \mathbb{R}^m$ are two expanding maps, then

$$g^\alpha(L'L) \geq g^\alpha(L') + g^\alpha(L). \tag{1.2}$$

The paper is organized as follows. In Section 2, we develop sub-additive thermodynamics formalism and prove the upper bounds of Hausdorff dimension of non-conformal repellers in [18] is exactly the unique root of the equation of sub-additive topological pressure. In Section 3, we consider the relation between sub-additive thermodynamics formalism defined in [4] and [2, 8], and we obtain for C^1 non-conformal repeller Λ, upper box dimension is bounded by a value which is the unique solution of the equation of sub-additive topological pressure. This is generalization of the result in [2]. In Section 4, we introduce the definition of average conformal repeller and give related results and the main theorem. In section 5, we develop super-additive thermodynamics formalism and variational principle for super-additive potential. In section 6, we give the proof of main result.

2 A sub-additive thermodynamics formalism

Let $f : X \to X$ be a continuous map. A set $E \subset X$ is called (n, ϵ) separated set with respect to f if $x, y \in E$ then $d_n(x, y) = \max_{0 \leq i \leq n-1} d(f^i x, f^i y) > \epsilon$. A sub-additive valuation on X is a sequence of continuous functions $\phi_n : M \to R$ such that

$$\phi_{m+n}(x) \leq \phi_n(x) + \phi_m(f^n(x)),$$

we denote it by $\mathcal{F} = \{\phi_n\}$.

In the following we will define the topological pressure of $\mathcal{F} = \{\phi_n\}$ with respect to f. We define

$$P_n(\mathcal{F}, \epsilon) = \sup\{\sum_{x \in E} \exp \phi_n(x) : E \quad \text{is a } (n, \epsilon) - \text{separated subset of } X\}.$$

Then the topological pressure of $\mathcal{F}$ is given by

$$P(f, \mathcal{F}) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log P_n(\mathcal{F}, \epsilon).$$

Let $\mathcal{M}(X)$ be the space of all Borel probability measures endowed with the weak* topology. Let $\mathcal{M}(X, f)$ denote the subspace of $\mathcal{M}(X)$ consisting of all f-invariant measures. For $\mu \in \mathcal{M}(X, f)$, let $h_\mu(f)$ denote the entropy of f with respect to μ, and let $\mathcal{F}_*(\mu)$ denote the following limit

$$\mathcal{F}_*(\mu) = \lim_{n \to \infty} \frac{1}{n} \int \phi_n d\mu.$$

The existence of the above limit follows from a sub-additive argument. We call $\mathcal{F}_*(\mu)$ the *Lyapunov exponent of* $\mathcal{F}$ *with respect to* μ since it describes the exponential growth speed of ϕ_n with respect to μ.

In [4], authors proved that the following variational principle

Theorem 2.1. *[4] Under the above general setting, we have*

$$P(f,\mathcal{F})=\sup\{h_\mu(T)+\mathcal{F}_*(\mu):\ \mu\in\mathcal{M}(X,f)\}.$$

In [2], Barreira used different way to introduce topological pressure for sub-additive potential and proved the variational principle if potential functions satisfies further condition.

Let M be a C^∞ Riemann manifold, $\dim M=m$. Let U be an open subset of M and let $f:U\to M$ be a C^1 map. Suppose $\Lambda\subset U$ is a compact invariant set on which f is expanding, that is, $f\Lambda=\Lambda$ and there is $k>1$ such that for all $x\in\Lambda$ and $v\in T_xM$,

$$\|D_xfv\|\geq k\|v\|,$$

where $\|.\|$ is the norm induced by an adapted Riemannian metric. Let $\mathcal{M}(f|_\Lambda)$ denote the all f invariant measures supported on Λ.

If $x\in\Lambda$, then $D_xf:T_xM\to T_{fx}M$ is a linear map. Denote the logarithms of the singular values of D_xf by

$$\lambda_1(x,f)\geq\lambda_2(x,f)\geq\cdots\geq\lambda_m(x,f)\geq\log k$$

and for $\alpha\in[0,m]$, write

$$g^\alpha(x,f)=g^\alpha(D_xf)=\sum_{i=m-[\alpha]+1}^{m}\lambda_i(x,f)+(\alpha-[\alpha])\lambda_{m-[\alpha]}(x,f).$$

Since f is C^1, the function $x\mapsto\lambda_i(x,f)$ and $x\mapsto g^\alpha(x,f)$ are all continuous.

In fact, $f\Lambda=\Lambda$ implies $f^n\Lambda=\Lambda$. f^n is also expanding on Λ. Let the logarithms of the singular value of D_xf^n be

$$\lambda_1(x,f^n)\geq\lambda_2(x,f^n)\geq\cdots\geq\lambda_m(x,f^n)\geq n\log k$$

and set

$$g^\alpha(x,f^n)=g^\alpha(D_xf^n)=\sum_{i=m-[\alpha]+1}^{m}\lambda_i(x,f^n)+(\alpha-[\alpha])\lambda_{m-[\alpha]}(x,f^n).$$

The functions $g^\alpha(\cdot,f^n)$ satisfy

$$g^\alpha(x,f^{n+l})\geq g^\alpha(x,f^n)+g^\alpha(f^n(x),f^l).$$

Define a sequence of functions $P_n:[0,m]\to R$ as follows:

$$P_n(\alpha)=P(f|_\Lambda,-\frac{1}{n}g^\alpha(\cdot,f^n)).$$

In [18], author proved that the following result:

Lemma 2.1. *[18] For every $\alpha \in [0, m]$, the following limit exists*

$$\lim_{n\to\infty} P_n(\alpha) = \inf_{n\in Z^+} P_n(\alpha).$$

Set $P^*(\alpha) = \lim_{n\to\infty} P_n(\alpha)$. Then P^* is a continuous and strictly decreasing on $[0, m]$.

Theorem 2.2. *[18] Let*

$$\mathcal{D}(f, \Lambda) = \max\{\alpha \in [0, m] : P^*(\alpha) \geq 0\}.$$

Then

$$\dim_H \Lambda \leq \mathcal{D}(f, \Lambda).$$

Remark 1. *By variational principle and Ruelle inequality, it has $P^*(m) \leq 0$. Since $P^*(0) = h(f|_\Lambda) > 0$, by Lemma 2.1, it follows that the equation $P^*(\alpha) = 0$ has an unique solution on $[0, m]$. By the definition, we have $\mathcal{D}(f, \Lambda)$ is the unique solution of the equation $P^*(\alpha) = 0$.*

In this paper, we first prove the following Proposition.

Proposition 2.1. *Suppose $\{\phi_n(x)\}$ be sub-additive continuous functions sequence on M. Let $\mathcal{F} = \{\phi_n\}$, then we have $P(f, \mathcal{F}) = \lim_{n\to\infty} P(f, \frac{\phi_n}{n})$*

Proof. The existence of limit $\lim_{n\to\infty} P(f, \frac{\phi_n}{n})$ can be found in [18].
First we prove that

$$P(f, \mathcal{F}) \leq \lim_{n\to\infty} P(f, \frac{\phi_n}{n}).$$

For a fixed m, let $n = ms + l, 0 \leq l < m$. From the subadditivity of $\{\phi_n\}$, we have

$$\phi_n(x) \leq \frac{1}{m}\sum_{j=0}^{m-1}\sum_{i=0}^{s-2} \phi_m(f^{im+j}(x)) + \frac{1}{m}\sum_{j=0}^{m-1}[\phi_j(x) + \phi_{m-j+l}(f^{(s-1)m+j}(x))].$$

Let $C_1 = \max_{i=1,\cdots,2m-1} \max_{x\in X} \phi_i(x)$. Then it has

$$\begin{aligned}\phi_n(x) &\leq \sum_{j=0}^{(sm+l)-1} \frac{1}{m}\phi_m(f^j(x)) - \frac{1}{m}\sum_{j=(s-1)m}^{sm-1} \phi_m(f^j(x)) + 2C_1 \\ &\leq \sum_{j=0}^{n-1} \frac{1}{m}\phi_m(f^j(x)) + 4C_1.\end{aligned}$$

Hence we have

$$\exp(\phi_n(x)) \leq \exp(\sum_{j=0}^{n-1} \frac{1}{m}\phi_m(f^j(x)) + 4C_1).$$

Thus

$$\begin{aligned} P_n(\mathcal{F},\epsilon) &= \sup\{\sum_{x\in E}\exp\phi_n(x) : E \text{ is a } (n,\epsilon)-\text{separated subset of } X\} \\ &\leq P_n(\frac{1}{m}\phi_m,\epsilon)\times\exp(4C_1). \end{aligned}$$

It implies

$$P(f,\mathcal{F}) \leq P(f,\frac{1}{m}\phi_m)\ .$$

From the arbitrary of $m\in Z^+$, we have

$$P(f,\mathcal{F}) \leq P(f,\frac{1}{m}\phi_m), \quad \text{for all } \ m\in Z^+.$$

Therefore

$$P(f,\mathcal{F}) \leq \lim_{n\to\infty} P(f,\frac{\phi_n}{n}).$$

Next, we prove that

$$P(f,\mathcal{F}) \geq \lim_{n\to\infty} P(f,\frac{\phi_n}{n}).$$

Since $f:\Lambda\to\Lambda$ is expanding map, $h_\mu(f)$ is an upper-semi continuous function from $\mathcal{M}(f|_\Lambda)$ to R. From variational principle of topological pressure [17], we have that for every $k\in Z^+$ there exists $\mu_{2^k}\in\mathcal{M}(f|_\Lambda)$ such that

$$P(f|_\Lambda,\frac{1}{2^k}\phi_{2^k}) = h_{\mu_{2^k}}(f) + \int_\Lambda \frac{1}{2^k}\phi_{2^k}d\mu_{2^k}.$$

Since $\mathcal{M}(f|_\Lambda)$ is compact, it implies that μ_{2^k} has a subsequence which converges to $\mu\in\mathcal{M}(f|_\Lambda)$. Without loss of generality, suppose that μ_{2^k} converges to μ. Using the subadditivity and invariant of μ_{2^k}, then we have for every $k\in\mathbb{N}$

$$h_{\mu_{2^k}}(f) + \int_\Lambda \frac{\phi_{2^k}(x)}{2^k}d\mu_{2^k} \leq h_{\mu_{2^k}}(f) + \int_\Lambda \phi_1(x)d\mu_{2^k}.$$

Furthermore for fixed $s\in\mathbb{N}$. If $k>s$, from the subadditivity and invariance of μ_{2^k}, it has

$$h_{\mu_{2^k}}(f) + \int_\Lambda \frac{\phi_{2^k}(x)}{2^k}d\mu_{2^k} \leq h_{\mu_{2^k}}(f) + \int_\Lambda \frac{\phi_{2^s}(x)}{2^s}d\mu_{2^k}.$$

Since $h_\mu(f)$ is a upper-semi continuous function, we have

$$\begin{aligned} \lim_{n\to\infty} P(f,\frac{\phi_n}{n}) &= \lim_{k\to\infty} P(f,\frac{\phi_{2^k}}{2^k}) \\ &= \lim_{k\to\infty}(h_{\mu_{2^k}}(f) + \int_\Lambda \frac{\phi_{2^k}(x)}{2^k}d\mu_{2^k}) \\ &\leq \lim_{k\to\infty}(h_{\mu_{2^k}}(f) + \int_\Lambda \frac{\phi_{2^s}(x)}{2^s}d\mu_{2^k}) \\ &\leq h_\mu(f) + \int_\Lambda \frac{\phi_{2^s}(x)}{2^s}d\mu. \end{aligned}$$

Since sequence $\{\int_\Lambda \phi_n(x)d\mu\}$ is sub-additive sequence, it has

$$\lim_{n\to\infty}\int_\Lambda \frac{\phi_n(x)}{n}d\mu = \inf_{n\geq 1}\{\int_\Lambda \frac{\phi_n(x)}{n}d\mu\}.$$

The arbitrariness of $s\in\mathbb{N}$ implies that

$$\lim_{n\to\infty} P(f,\frac{\phi_n}{n}) \leq h_\mu(f) + \lim_{s\to\infty}\int_\Lambda \frac{\phi_{2^s}}{2^s}(x)d\mu.$$

Hence by variational principle of the sub-additive topological pressure in [4], we have

$$\lim_{n\to\infty} P(f,\frac{\phi_n}{n}) \leq h_\mu(f) + \lim_{s\to\infty}\int_\Lambda \frac{\phi_{2^s}}{2^s}(x)d\mu \leq P(f,\mathcal{F}).$$

This completes the proof of Proposition 2.1. □

Theorem 2.3. *Let $\mathcal{F}(\alpha) = \{-g^\alpha(\cdot, f^n)\}$. Then we have $P(f,\mathcal{F}(\alpha)) = P^*(\alpha)$.*

Proof. For a fixed α, let $\phi_n(x) = -g^\alpha(x, f^n)$, then it is sub-additive continuous sequence on Λ. By Proposition 2.1 for $\mathcal{F}(\alpha) = \{-g^\alpha(x, f^n)\}$, we have

$$P(f,\mathcal{F}(\alpha)) = \lim_{n\to\infty} P(f,\frac{\phi_n}{n}) = \lim_{n\to\infty} P(f, -\frac{1}{n}g^\alpha(\cdot, f^n)) = P^*(\alpha).$$

□

Theorem 2.4. *Let $\mathcal{F}(\alpha) = \{-g^\alpha(\cdot, f^n)\}$, then we have $P(f,\mathcal{F}(\alpha))$ is continuous and strictly monotone decreasing on $\alpha\in[0,m]$. Thus $P(f,\mathcal{F}(\alpha)) = 0$ has only unique solution in $[0,m]$.*

Proof. Let $\phi_n(\alpha, x) = -g^\alpha(x, f^n)$. If $\alpha_1,\ \alpha_2\in[0,m],\ \alpha_1<\alpha_2$, then for all $n\in\mathbb{N}$, it has

$$(\alpha_1-\alpha_2)n\log k \geq -\phi_n(\alpha_2,x) - (-\phi_n(\alpha_1,x)) \geq (\alpha_1-\alpha_2)n\log\|f\|.$$

Thus for $\forall\epsilon>0$,

$$e^{(\alpha_1-\alpha_2)n\log k}\times P_n(\mathcal{F}(\alpha_1),\epsilon) \leq P_n(\mathcal{F}(\alpha_2),\epsilon) \leq P_n(\mathcal{F}(\alpha_1),\epsilon)\times e^{(\alpha_1-\alpha_2)n\log\|f\|}.$$

It implies that

$$(\alpha_1-\alpha_2)\log\|f\| + P(f,\mathcal{F}(\alpha_1)) \leq P(f,\mathcal{F}(\alpha_2)) \leq P(f,\mathcal{F}(\alpha_1)) + (\alpha_1-\alpha_2)\log k.$$

Therefore $P(f,\mathcal{F}(\alpha))$ is continuous and strictly monotone decreasing on $\alpha\in[0,m]$.

One hand, $P(f,\mathcal{F}(0)) = h_{top}(f) > 0$, and on the other hand, by Ruelle inequality [14] and Theorem 2.1, it has $P(f,\mathcal{F}(m))\leq 0$. Therefore $P(f,\mathcal{F}(\alpha)) = 0$ has an unique solution in $[0,m]$. □

Remark 2. *Theorem 2.4 can be deduced from Theorem 2.3 and Lemma 2.1. But for the completeness, we include a different proof.*

Corollary 2. *$\mathcal{D}(\Lambda, f)$ is the unique solution of equation $P(f, \mathcal{F}(\alpha)) = 0$.*

Proof. The proof can be deduced from Theorem 2.3 and Remark 1. □

Lemma 2.2. *For a fixed $n \in \mathbb{N}$, $P_n(\alpha) = P(f, -\frac{1}{n}g^{\alpha}(\cdot, f^n))$ is a continuous and monotone decreasing function on $\alpha \in [0, m]$.*

Proof. The proof is analogous to the proof of Theorem 2.4. □

By Ruelle-Margulis inequality and variational principle in [17], it has $P_n(m) = P(f, -\frac{1}{n}g^{m}(\cdot, f^n)) \leq 0$. Since $P_n(0) = h(f|_\Lambda) > 0$, by Lemma 2.2, it follows that equation $P_n(\alpha) = 0$ has an unique solution. Denote it by α_n. Then we have the following proposition.

Theorem 2.5.

$$\inf_{n\in\mathbb{N}} \alpha_n = \mathcal{D}(\Lambda, f)$$

Proof. Without loss of generality, we suppose that $\lim\limits_{n\to\infty} \alpha_n = \alpha^* = \inf_{n\in\mathbb{N}} \alpha_n$. Otherwise we can take a subsequence which converges to α^*.

Since

$$\begin{aligned} |P(f, -\frac{1}{n}g^{\alpha^*}(\cdot, f^n)) - P(f, -\frac{1}{n}g^{\alpha_n}(\cdot, f^n))| &\leq \| - \frac{1}{n}g^{\alpha^*}(\cdot, f^n) + \frac{1}{n}g^{\alpha_n}(\cdot, f^n)\| \\ &\leq |\alpha^* - \alpha_n|\|Df\|, \end{aligned}$$

we have

$$\begin{aligned} P(f, \mathcal{F}(\alpha^*)) &= \lim_{n\to\infty} P(f, -\frac{1}{n}g^{\alpha^*}(\cdot, f^n)) \\ &= \lim_{n\to\infty} P(f, -\frac{1}{n}g^{\alpha_n}(\cdot, f^n)) = 0. \end{aligned}$$

By Corollary 2, we have

$$\mathcal{D}(f, \Lambda) = \alpha^* = \inf_{n\in\mathbb{N}} \alpha_n.$$

□

Now for a fixed $n \in \mathbb{N}$, we consider the equation

$$\tilde{P}_n(\alpha) = P(f^n|_\Lambda, -g^{\alpha}(\cdot, f^n)) = 0.$$

It is easy to prove that $\tilde{P}_n(\alpha)$ is continuous and strictly decreasing on $[0, m]$.

$$\tilde{P}_n(0) = h_{top}(f^n|_\Lambda) = nh_{top}(f|_\Lambda) \geq 0$$

and

$$\tilde{P}_n(m) = nP(f|_\Lambda, -\log|Jac(D_x f)|) \leq 0.$$

Hence the equation $\tilde{P}_n(\alpha) = 0$ has a unique solution, which we denoted by D_n. Applying Lemma 1 in [18] to the expanding map f^n yield $\dim_H \Lambda \leq D_n$. So $\dim_H \Lambda \leq \inf_{n\in Z^+} D_n$. It was proved in [18] that

$$\inf_{n\in Z^+} D_n \leq \mathcal{D}(f,\Lambda).$$

Next we want to prove the reverse inequality, that is say

$$\mathcal{D}(f,\Lambda) \leq \inf_{n\in Z^+} D_n.$$

In order to prove the inequality as above, we firstly prove the following theorem

Proposition 2.2. *Suppose $\{\phi_n(x)\}$ be sub-additive continuous sequence on M. Let $\mathcal{F} = \{\phi_n\}$, then we have $P(f,\mathcal{F}) = \lim_{k\to\infty} \frac{1}{k} P(f^k, \phi_k)$.*

Proof. For a fixed $k \in \mathbb{N}$. It is well known that if $E \subset M$ is an (n,ϵ) separated set of f^k, then E is an (nk,ϵ) separated set of f. By the definition

$$\begin{aligned} P(f^k,\phi_k) &= \lim_{\epsilon\to\infty} \limsup_{n\to\infty} \frac{1}{n} \log\sup\{\sum_{x\in E} \exp(\hat{S}_n\phi_k(x)) \\ &: \ E \text{ is a } (n,\epsilon) \text{ separated set of } f^k\}, \end{aligned}$$

where

$$(\hat{S}_n\phi_k(x)) = \phi_k(x) + \phi_k(f^k x) + \cdots + \phi_k(f^{(n-1)k}x).$$

Hence for a fixed $m < k$, let $k = mq + r$ and $C = \max_{x\in M} \max_{i=1,\cdots,2m} \phi_i(x)$, the subadditivity of ϕ_n implies that

$$\begin{aligned} \phi_k(x) &\leq \frac{1}{m}\sum_{j=0}^{m-1}\sum_{i=0}^{q-2} \phi_m(f^{im+j}(x)) + \frac{1}{m}\sum_{j=0}^{m-1}[\phi_j(x) + \phi_{m-j+l}(f^{(q-1)m+j}(x))] \\ &\leq \sum_{i=0}^{k-1} \frac{1}{m}\phi_m(f^i(x)) + 4C. \end{aligned}$$

Thus for $1 \leq j \leq n-1$, we have

$$\phi_k(f^{kj}(x)) \leq \sum_{i=0}^{k-1} \frac{1}{m}\phi_m(f^i(f^{kj}(x)) + 4C.$$

Hence

$$\begin{aligned} \hat{S}_n\phi_k(x) &= \phi_k(x) + \phi_k(f^k x) + \cdots + \phi_k(f^{(n-1)k}x) \\ &\leq \sum_{i=0}^{nk-1} \frac{1}{m}\phi_m(f^i(x)) + 4nC \\ &= S_{nk}(\frac{1}{m}\phi_m)(x) + 4nC. \end{aligned}$$

It gives that

$$P_n(f^k, \phi_k, \epsilon) \;\; \leq \;\; P_{nk}(f, \frac{1}{m}\phi_m, \epsilon) \times e^{4nC}.$$

Thus

$$\begin{aligned} P(f^k, \phi_k) \;\; &\leq \;\; kP(f, \frac{1}{m}\phi_m) + \lim_{n\to\infty} \frac{1}{n} \log e^{4nC} \\ &= \;\; kP(f, \frac{1}{m}\phi_m) + 4C. \end{aligned}$$

Therefore

$$\lim_{k\to\infty} \frac{1}{k} P(f^k, \phi_k) \leq P(f, \frac{1}{m}\phi_m) \quad \text{for all } m \in Z^+.$$

By Theorem 2.1, it has

$$\lim_{k\to\infty} \frac{1}{k} P(f^k, \phi_k) \leq \lim_{m\to\infty} P(f, \frac{1}{m}\phi_m) = P(f, \mathcal{F}).$$

Next we prove that

$$P(f, \mathcal{F}) \leq \lim_{k\to\infty} \frac{1}{k} P(f^k, \phi_k).$$

For a fixed $k \in \mathbb{N}$, let $n = km + r, \;\; 0 \leq r < k$, and let $C = \max_{x\in M} \max_{1\leq i\leq k} \phi_i(x)$. For $\forall\epsilon > 0$, by the uniformly continuity of f, there exists $\delta > 0$ such that if $E \subset M$ is an (n, ϵ) separated set of f, then E is an (m, δ) separated set of f^k and $\delta \to 0$ when $\epsilon \to 0$. Using the subadditivity of ϕ_n, we have

$$\phi_n(x) \leq \phi_k(x) + \phi_k(f^k(x)) + \cdots + \phi_k(f^{(m-1)k}(x)) + \phi_r(f^{mk}(x)).$$

Thus

$$P_n(f, \mathcal{F}, \epsilon) \leq P_m(f^k, \phi_k, \delta) \times e^C.$$

Hence

$$P(f, \mathcal{F}, \epsilon) \leq \frac{1}{k} P(f^k, \phi_k, \delta).$$

It gives that

$$P(f, \mathcal{F}) \leq \frac{1}{k} P(f^k, \phi_k).$$

From the arbitrary of k, we have that

$$P(f, \mathcal{F}) \leq \lim_{k\to\infty} \frac{1}{k} P(f^k, \phi_k).$$

$\square$

Corollary 3. *Let $\mathcal{F}(\alpha) = \{-g^\alpha(\cdot, f^k)\}$, then we have*

$$P(f, \mathcal{F}(\alpha)) = \lim_{k\to\infty} \frac{1}{k} P(f^k, -g^\alpha(\cdot, f^k)).$$

Proof. Fixed α, let $\phi_k(x) = -g^\alpha(x, f^k)$. Using Theorem 2.2 for $\mathcal{F}(\alpha) = \{-g^\alpha(x, f^k)\}$, we get

$$P(f, \mathcal{F}(\alpha)) = \lim_{k\to\infty} \frac{1}{k} P(f^k, -g^\alpha(\cdot, f^k)).$$

□

Theorem 2.6.

$$\inf_{n\in\mathbb{N}} D_n = \mathcal{D}(\Lambda, f)$$

Proof. Without loss of generality, we suppose that $\lim\limits_{n\to\infty} D_n = \beta^* = \inf_{n\in\mathbb{N}} D_n$. Otherwise we can take a subsequence which converges to β^*. Since

$$\begin{aligned} |\frac{1}{k}P(f^k, -g^{\beta^*}(\cdot, f^k)) - \frac{1}{k}P(f^k, -g^{D_k}(\cdot, f^k))| &\leq \| -\frac{1}{k}g^{\beta^*}(\cdot, f^k) + \frac{1}{k}g^{D_k}(\cdot, f^k)\| \\ &\leq |\beta^* - D_k|\|Df\|, \end{aligned}$$

we have

$$\begin{aligned} P(f, \mathcal{F}(\beta^*)) &= \lim_{k\to\infty} \frac{1}{k} P(f^k, -g^{\beta^*}(\cdot, f^k)) \\ &= \lim_{k\to\infty} \frac{1}{k} P(f^k, -g^{D_k}(\cdot, f^k)) = 0. \end{aligned}$$

Thus

$$\mathcal{D}(f, \Lambda) = \beta^* = \inf_{k\in\mathbb{N}} D_k.$$

□

In this section, we have proven that for C^1 non-conformal repeller Λ, $\mathcal{D}(f, \Lambda)$, which is the unique solution of equation $P(f, \mathcal{F}(\alpha)) = 0$, is the upper bounds of Hausdorff dimension of Λ. This is a generalization of the classical result which for $C^{1+\gamma}$ conformal repeller Λ, $dim_H\Lambda$ is given by the unique solution of the equation $P(f|_\Lambda, -\alpha \log \|D_x f\|) = 0$. Moreover, we prove that

$$\mathcal{D}(f, \Lambda) = \inf_{n\in\mathbb{N}} D_k = \inf_{n\in\mathbb{N}} \alpha_k,$$

where for each $n \in \mathbb{N}$, D_n and α_n are the unique solutions of equations $\tilde{P}_n(\alpha) = 0$ and $P_n(\alpha) = 0$ respectively.

3 Other results of upper bounded estimates of dimension for repeller

Let us first recall Falconer's definition of topological pressures for sub-additive potentials on mixing repellers. Without loss of generality, we only consider one-sided sub-shift spaces of finite type rather than mixing repellers.

Let (Σ_A, σ) be a one-sided sub-shift space over an alphabet $\{1, \ldots, m\}$, where $m \geq 2$. As usual Σ_A is endowed with the metric $d(x, y) = m^{-n}$ where $x = (x_k),\ y = (y_k)$ and n is the smallest of the k such that $x_k \neq y_k$. For any admissible string $I = i_1 \ldots i_n$ of length n over the letters $\{1, \ldots, m\}$, denote $[I] = \{(x_i) \in \Sigma :\ x_j = i_j \text{ for } 1 \leq j \leq n\}$. The $[I]$ is called an n-th cylinder in Σ_A.

Let $\mathcal{F}$ be a sub-additive family of continuous potentials define on Σ. Falconer defined the topological pressure of $\mathcal{F}$ by

$$FP(\sigma, \mathcal{F}) = \lim_{n\to\infty} \frac{1}{n} \log FP_n(\sigma, \mathcal{F}) \quad \text{and} \quad FP_n(\sigma, \mathcal{F}) = \sum_{[I]} \sup_{x\in[I]} e^{\phi_n(x)},$$

where the summation is taken over the collection of all nth cylinders $[I]$'s.

It is not so hard to see that in this special case, $FP_n(\sigma, \mathcal{F}) = P_n(\sigma, \mathcal{F}, 1/m)$, and $P_n(\sigma, \mathcal{F}, m^{-k}) = FP_{n+k-1}(\sigma, \mathcal{F})$ for all $k \in \mathbb{N}$. This implies $FP(\sigma, \mathcal{F})$ is equivalent to our definition $P(\sigma, \mathcal{F})$.

Now let us turn to Barreira's approach in defining pressures for sub-additive potentials via open covers.

As in the previous sections, let f be a continuous map acting on a compact metric space (X, d). Let $\mathcal{F} = \{\phi_n\}_{n=1}^{\infty}$ be a family of sub-additive continuous functions defined on X. Suppose $\mathcal{U}$ is a finite open cover of the space X. For $n \geq 1$ we denote by $\mathcal{W}_n(\mathcal{U})$ the collection of strings $\mathbf{U} = U_1 \ldots U_n$ with $U_i \in \mathcal{U}$. For $\mathbf{U} \in \mathcal{W}_n(\mathcal{U})$ we call the integer $m(\mathbf{U}) = n$ the length of $\mathbf{U}$ and define

$$\begin{aligned} X(\mathbf{U}) &= U_1 \cap f^{-1}U_2 \cap \ldots \cap f^{-(n-1)}U_n \\ &= \left\{x \in X :\ f^{j-1}x \in U_j \text{ for } j = 1, \ldots, n\right\}. \end{aligned}$$

We say that $\Gamma \subset \bigcup_{n\geq 1} \mathcal{W}_n(\mathcal{U})$ covers X if $\bigcup_{\mathbf{U}\in\Gamma} X(\mathbf{U}) = X$. For each $\mathbf{U} \in \mathcal{W}_n(\mathcal{U})$, we write $e^{\phi(\mathbf{U})} = \sup_{x\in X(\mathbf{U})} e^{\phi_n(x)}$ when $X(\mathbf{U}) \neq \emptyset$ and $e^{\phi(\mathbf{U})} = -\infty$ otherwise. For $s \in \mathbb{R}$, define

$$M(f, s, \mathcal{F}, \mathcal{U}) = \lim_{n\to\infty} \inf\{\sum_{\mathbf{U}\in\Gamma} e^{-sm(\mathbf{U})} e^{\phi(\mathbf{U})}\},$$

where the infimum is taken over all $\Gamma \subset \bigcup_{j\geq n} \mathcal{W}_j(\mathcal{U})$ that covers X. Likewise, we define

$$\underline{M}(f, s, \mathcal{F}, \mathcal{U}) = \liminf_{n\to\infty} \inf\{\sum_{\mathbf{U}\in\Gamma} e^{-sm(\mathbf{U})} e^{\phi(\mathbf{U})}\},$$

$$\overline{M}(f, s, \mathcal{F}, \mathcal{U}) = \limsup_{n\to\infty} \inf\{\sum_{\mathbf{U}\in\Gamma} e^{-sm(\mathbf{U})} e^{\phi(\mathbf{U})}\},$$

where the infimum is taken over all $\Gamma \subset \mathcal{W}_n(\mathcal{U})$ that covers X. Define

$$P^\star(f,\mathcal{F},\mathcal{U}) = \inf\{s:\ M(f,s,\mathcal{F},\mathcal{U})=0\} = \sup\{s:\ M(f,s,\mathcal{F},\mathcal{U})=+\infty\},$$

$$\underline{CP^\star}(f,\mathcal{F},\mathcal{U}) = \inf\{s:\ \underline{M}(f,s,\mathcal{F},\mathcal{U})=0\} = \sup\{s:\ \underline{M}(f,s,\mathcal{F},\mathcal{U})=+\infty\},$$

$$\overline{CP^\star}(f,\mathcal{F},\mathcal{U}) = \inf\{s:\ \overline{M}(f,s,\mathcal{F},\mathcal{U})=0\} = \sup\{s:\ \overline{M}(f,s,\mathcal{F},\mathcal{U})=+\infty\}.$$

Define

$$P^\star(f,\mathcal{F}) = \liminf_{\mathrm{diam}(\mathcal{U})\to 0} P^\star(f,\mathcal{F},\mathcal{U}),$$

$$\underline{CP^\star}(f,\mathcal{F}) = \liminf_{\mathrm{diam}(\mathcal{U})\to 0} \underline{CP^\star}(f,\mathcal{F},\mathcal{U}),$$

$$\overline{CP^\star}(f,\mathcal{F}) = \liminf_{\mathrm{diam}(\mathcal{U})\to 0} \overline{CP^\star}(f,\mathcal{F},\mathcal{U}).$$

Barreira named $P^\star(f,\mathcal{F})$ the topological pressure, $\underline{CP^\star}(f,\mathcal{F})$ and $\overline{CP^\star}(f,\mathcal{F})$ the lower and upper topological pressures of $\mathcal{F}$.

Now we consider the connection between $P^\star(f,\mathcal{F})$ and $P(f,\mathcal{F})$. In [4], we prove the following equality.

Proposition 3.1. *Assume the topological entropy $h(f) < \infty$ and the entropy map $\mu \mapsto h_\mu(f)$ is upper semi-continuous. Then $P^\star(f,\mathcal{F}) = P(f,\mathcal{F})$.*

Theorem 3.1. *Let M be a C^∞ Riemann manifold and $f : M \to M$ be a C^1 map. Suppose $\Lambda \subset M$ is a compact invariant set on which f is expanding. Then*

$$P^\star(f,\mathcal{F}(\alpha)) = P(f,\mathcal{F}(\alpha)).$$

Proof. Since $\Lambda \subset M$ is a compact invariant set on which f is expanding, it has measure theoretical entropy $h_\mu(f|_\Lambda)$ is an upper semi-continuous map in $\mathcal{M}(f|_\Lambda)$. By proposition 3.1, it has

$$P^\star(f,\mathcal{F}(\alpha)) = P(f,\mathcal{F}(\alpha)).$$

□

In [1], Barreira proved that if Λ is a repeller of a $C^{1+\gamma}$ map, for some $\gamma > 0$ and f is $\gamma-$bunched on Λ, then $\overline{\dim}_B \le t^*$, where t^* is the unique number of equation $P^\star(f,\mathcal{F}(\alpha)) = 0$. In [1], $\gamma-$bunched condition and $C^{1+\gamma}$ were used to show that it is reasonable to define $P^\star(f,\mathcal{F}(\alpha))$.

Corollary 4. *Let M be a C^∞ Riemann manifold and $f : M \to M$ be a C^1 map. Suppose $\Lambda \subset M$ is a compact invariant set on which f is expanding. Then*

$$dim_B\Lambda \le \mathcal{D}(\Lambda,f) \ \ and\ dim_H\Lambda \le \mathcal{D}(\Lambda,f),$$

where $\mathcal{D}(\Lambda,f)$ is the unique solution of equation $P(f,\mathcal{F}(\alpha)) = 0$.

Proof. By Theorem 3.1, we have that if Λ is a repeller of a C^1 map, then we can define $P^{\star}(f, \mathcal{F}(\alpha))$ and prove that it is coincidence with $P(f, \mathcal{F}(\alpha))$. It is proved in [1] that $\overline{\dim}_B \leq t^*$, where t^* is the unique solution of equation $P^{\star}(f, \mathcal{F}(\alpha)) = 0$. Thus we have that $t^* = \mathcal{D}(\Lambda, f)$ which is the unique solution of equation $P(f, \mathcal{F}(\alpha)) = 0$. Therefore we also have the inequality for box dimension. □

Remark 3. *In [18], Zhang posed a problem whether $\mathcal{D}(\Lambda, f)$ is the upper bound of box dimension of Λ. Corollary as above gives an affirmative answer to the problem. Moreover, our result shows that the subadditive thermodynamic formalism can be apply. In fact we have proven that if Λ is a repeller of a C^1 map, then the upper bounds of Hausdorff dimension of Λ by Barreira in [1], Falconer in [8] and Zhang in [18] are coincide. This unifies their results and it also shows that bunched condition in [1] and [8] is unnecessary. Our result also gives an affirmative answer to problem posed by K.Simon in [15] about an upper bound without assuming the 1-bunched property.*

4 Average conformal repeller

Let M be a C^∞ Riemann manifold, $\dim M = d$. Let U be an open subset of M and let $f : U \to M$ be a C^1 map. Suppose $\Lambda \subset U$ is a compact expanding invariant set. Let $\mathcal{E}(f)$ denote the all ergodic invariant measure supported on Λ respectively. By the Oseledec multiplicative ergodic theorem, for any $\mu \in \mathcal{E}(f)$, we can define Lyapunov exponents $\lambda_1(\mu) \leq \lambda_2(\mu) \leq \cdots \leq \lambda_d(\mu)$.

Definition 4.1. *An invariant repeller is called average conformal if for any $\mu \in \mathcal{E}(f)$, $\lambda_1(\mu) = \lambda_2(\mu) = \cdots = \lambda_d(\mu) > 0$.*

It is obvious that a conformal repeller is an average conformal repeller, but reverse isn't true.

Next we will give main theorem.

Theorem 4.1. (Main Theorem) *Let f be C^1 dynamical system and Λ be an average conformal repeller, then the Hausdorff dimension of Λ is zero t_0 of $t \mapsto P(-t\mathcal{F})$, where*

$$\mathcal{F} = \{\log(m(Df^n(x)), x \in \Lambda, n \in \mathbb{N}\}. \tag{4.3}$$

where $m(A) = \|A^{-1}\|^{-1}$

The proof will be given in section 6.

Theorem 4.2. *If Λ be an average conformal repeller, then*

$$\lim_{n\to\infty} \frac{1}{n}(\log \|Df^n(x)\| - \log m(Df^n(x))) = 0$$

uniformly on Λ.

Proof. Let

$$F_n(x) = \log \|Df^n(x)\| - \log m(Df^n(x)),\ n \in \mathbb{N},\ x \in \Lambda.$$

It is obviously that the sequence $\{F_n(x)\}$ is a non-negative subadditive function sequence. That is say

$$F_{n+m}(x) \le F_n(x) + F_m(f^n(x)),\ \ x \in \Lambda.$$

Suppose (4.3) is not true, then there exists $\epsilon_0 > 0$, for any $k \in \mathbb{N}$, there exits $n_k \ge k$ and $x_{n_k} \in \Lambda$ such that

$$\frac{1}{n_k} F_{n_k}(x_{n_k}) \ge \epsilon_0.$$

Define measures

$$\mu_{n_k} = \frac{1}{n_k} \sum_{i=0}^{n_k - 1} \delta_{f^i(x_{n_k})}.$$

Compactness of $\mathcal{P}(f)$ implies there exists a subsequence of μ_{n_k} that converges to measure μ. Without loss of generality, we suppose that $\mu_{n_k} \to \mu$. It is well known that μ is f -invariant. Therefore $\mu \in \mathcal{M}(f)$.

For a fixed m, we have

$$\lim_{k\to\infty} \int_M \frac{1}{m} F_m(x_{n_k}) d\mu_{n_k} = \int_M \frac{1}{m} F_m(x_{n_k}) d\mu.$$

It implies

$$\lim_{k\to\infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} \frac{1}{m} F_m(f^i(x_{n_k})) = \int_M \frac{1}{m} F_m(x) d\mu.$$

For a fixed m, let $n_k = ms + l, 0 \le l < m$. The sub-additivity of $\{F_n\}$ implies that for $j = 0, \cdots m-1$,

$$\begin{aligned} F_{n_k}(x_{n_k}) \ \ &\le \ \ F_j(x_{n_k}) + F_m(f^j(x_{n_k}) + \cdots + F_m(f^{m(s-2)} f^j(x_{n_k})) \\ &\quad + F_{m-j+l}(f^{m(s-1)} f^j(x_{n_k})) \end{aligned}$$

Summing j from 0 to $m-1$, we get

$$\begin{aligned} F_{n_k}(x_{n_k}) \ \ &\le \ \ \frac{1}{m} \sum_{j=0}^{m-1} \sum_{i=0}^{s-2} F_m(f^{im+j}(x_{n_k})) \\ &+ \ \ \frac{1}{m} \sum_{j=0}^{m-1} [F_j(x_{n_k}) + F_{m-j+l}(f^{(s-1)m+j}(x_{n_k}))] \end{aligned}$$

Let $C_1 = \max_{i=1,\cdots,2m-1} \max_{x\in\Lambda} F_i(x)$.

$$\begin{aligned} F_{n_k}(x) &\leq \sum_{j=0}^{(sm+l)-1} \frac{1}{m} F_m(f^j(x)) - \frac{1}{m} \sum_{j=(s-1)m}^{sm-1} F_m(f^j(x)) + 2C_1 \\ &\leq \sum_{j=0}^{n_k-1} \frac{1}{m} F_m(f^j(x)) + 4C_1. \end{aligned}$$

Hence we have

$$\lim_{k\to\infty} \frac{1}{n_k} F_{n_k}(x) \leq \lim_{k\to\infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} \frac{1}{m} F_m(f^i(x)) = \int_M \frac{1}{m} F_m(x) d\mu.$$

The arbitrariness of $m \in \mathbb{N}$ implies that

$$\lim_{k\to\infty} \frac{1}{n_k} F_{n_k}(x) \leq \frac{1}{m} \int_M F_m(x) d\mu, \ \ \forall m \in \mathbb{N}.$$

Hence

$$\lim_{m\to\infty} \frac{1}{m} \int_M F_m(x) d\mu \geq \epsilon_0 > 0.$$

Then ergodic decomposition theorem [17] implies that there exists $\tilde{\mu} \in \mathcal{E}(f)$ such that

$$\lim_{m\to\infty} \frac{1}{m} \int_M F_m(x) d\tilde{\mu} \geq \epsilon_0 > 0.$$

On the other hand, from Oseledec theorem and Kingman's subadditive ergodic theorem, we have $\lim\limits_{m\to\infty} \frac{1}{m} \int_M \log \|Df^m(x)\| d\tilde{\mu} = \lambda_d(\tilde{\mu})$ and $\lim\limits_{m\to\infty} \frac{1}{m} \int_M \log m(Df^m(x)) d\tilde{\mu} = \lambda_1(\tilde{\mu})$. Therefore

$$\lambda_d(\tilde{\mu}) - \lambda_1(\tilde{\mu}) \geq \epsilon_0.$$

This gives a contradiction to assumption of average conformal. □

5 Super-additive variational principle

In this section, we first give the definition of super-additive topological pressure. Then we prove the variational principle for special super-additive potential.

Let $f : X \to X$ be a continuous map. A set $E \subset X$ is called (n, ϵ) separated set with respect to f if $x, y \in E$ then $d_n(x, y) = \max_{0\leq i\leq n-1} d(f^i x, f^i y) > \epsilon$. A super-additive valuation on X is a sequence of functions $\varphi_n : M \to R$ such that

$$\varphi_{m+n}(x) \geq \varphi_n(x) + \varphi_m(f^n(x)),$$

we denote it by $\mathcal{F} = \{\varphi_n\}$.

In the following we will define the topological pressure of $\mathcal{F} = \{\varphi_n\}$ with respect to f. We define

$$P_n^*(\mathcal{F}, \epsilon) = \sup\{\sum_{x\in E} \exp \varphi_n(x) : E \quad \text{is a } (n, \epsilon) - \text{separated subset of } X\}.$$

Then the topological pressure of $\mathcal{F}$ is given by

$$P^*(f, \mathcal{F}) = \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log P_n(\mathcal{F}, \epsilon).$$

For every $\mu \in \mathcal{M}(X, f)$, let $\mathcal{F}_*(\mu)$ denote the following limit

$$\mathcal{F}_*(\mu) = \lim_{n\to\infty} \frac{1}{n} \int \varphi_n d\mu.$$

The existence of the above limit follows from a super-additive argument. We call $\mathcal{F}_*(\mu)$ the *Lyapunov exponent of $\mathcal{F}$ with respect to μ* since it describes the exponential growth speed of φ_n with respect to μ.

Theorem 5.1. *Let f be C^1 dynamical system and Λ be an average conformal repeller, and $\mathcal{F} = \{\varphi_n(x)\} = \{-t \log \|Df^n(x)\|\}$ for $t \geq 0$ be a super-additive function sequence. Then we have*

$$P^*(f, \mathcal{F}) = \sup\{h_\mu(T) + \mathcal{F}_*(\mu) : \ \mu \in \mathcal{M}(X, f)\}.$$

Proof. First we prove that for any $m \in \mathbb{N}$

$$P^*(f, \mathcal{F}) \geq P(f, \frac{\varphi_m}{m}).$$

For a fixed m, let $n = ms + l, 0 \leq l < m$. From the sup-additivity of $\{\varphi_n\}$, we have

$$\varphi_n(x) \geq \frac{1}{m} \sum_{j=0}^{m-1} \sum_{i=0}^{s-2} \varphi_m(f^{im+j}(x)) + \frac{1}{m} \sum_{j=0}^{m-1} [\varphi_j(x) + \varphi_{m-j+l}(f^{(s-1)m+j}(x))].$$

Let $C_1 = \min_{i=1,\cdots,2m-1} \min_{x\in X} \varphi_i(x)$. Then it has

$$\begin{aligned}
\varphi_n(x) &\geq \sum_{j=0}^{(sm+l)-1} \frac{1}{m} \varphi_m(f^j(x)) - \frac{1}{m} \sum_{j=(s-1)m}^{sm-1} \varphi_m(f^j(x)) + 2C_1 \\
&\geq \sum_{j=0}^{n-1} \frac{1}{m} \varphi_m(f^j(x)) + 4C_1.
\end{aligned}$$

Hence we have

$$\exp(\varphi_n(x)) \geq \exp(\sum_{j=0}^{n-1} \frac{1}{m}\varphi_m(f^j(x)) + 4C_1).$$

Thus

$$\begin{aligned} P_n^*(\mathcal{F}, \epsilon) &= \sup\{\sum_{x\in E} \exp \varphi_n(x) : E \text{ is a } (n,\epsilon) - \text{separated subset of } X\} \\ &\geq P_n(\frac{1}{m}\varphi_m, \epsilon) \times \exp(4C_1). \end{aligned}$$

It implies

$$P^*(f, \mathcal{F}) \geq P(f, \frac{1}{m}\varphi_m) \ .$$

From the arbitrary of $m \in Z^+$, we have

$$P^*(f, \mathcal{F}) \geq P(f, \frac{1}{m}\varphi_m), \quad \text{for all } \ m \in Z^+.$$

By the variational principle in [17], for every $\mu \in \mathcal{M}(f)$, we have

$$P^*(f, \mathcal{F}) \geq P(f, \frac{1}{m}\varphi_m) \geq h_\mu(f) + \int_M \frac{1}{m}\varphi_m(x)d\mu, \ \ \forall m \in \mathbb{N}.$$

Hence we have for every $\mu \in \mathcal{M}(f)$

$$P^*(f, \mathcal{F}) \geq h_\mu(f) + \lim_{m\to\infty} \int_M \frac{1}{m}\varphi_m(x)d\mu.$$

Therefore

$$P^*(f, \mathcal{F}) \geq \sup\{h_\mu(f) + \lim_{m\to\infty} \int_M \frac{1}{m}\varphi_m(x)d\mu, \ \ \mu \in \mathcal{M}(f)\}$$

Let $\Phi_n(x) = -t \log m(Df^n(x))$ for $t \geq 0$. Then it is sub-additive. By the theorem in [4], we have

$$P(f, \{\Phi_n\}) = \sup\{h_\mu(f) + \lim_{m\to\infty} \int_M \frac{1}{m}\Phi_m(x)d\mu, \ \ \mu \in \mathcal{M}(f)\}$$

By the definitions, $-t \log m(Df^n(x)) \geq -t \log \|Df^n(x)\|$ for $t \geq 0$ implies that

$$P^*(f, \mathcal{F}) \leq P(f, \{\Phi_n\}).$$

Theorem 4.3 implies that for any $\mu \in \mathcal{M}(f)$, it has

$$\lim_{m\to\infty} \int_M \frac{1}{m}\Phi_m(x)d\mu = \lim_{m\to\infty} \int_M \frac{1}{m}\varphi_m(x)d\mu.$$

Therefore

$$P^*(f, \mathcal{F}) = \sup\{h_\mu(f) + \lim_{m\to\infty} \int_M \frac{1}{m}\Phi_m(x)d\mu, \ \ \mu \in \mathcal{M}(f)\}.$$

This completes the proof of theorem. $\square$

6 The proof of main theorem

In this section, we will give the proof of main theorem. First we state some known results.

In [1], Barreira prove the following theorem.

Theorem 6.1. *If f is a C^1 expanding map and Λ is a repeller, then*

$$s_1 \leq dim_H \Lambda \leq \underline{dim}_B \Lambda \leq \overline{dim}_B \Lambda \leq t_1$$

where s_1 and t_1 are the unique roots of the Bowen's equations $P(f, -t \log \|Df(x)\|) = 0$ and $P(f, -t \log m(Df(x))) = 0$ respectively.

Since Λ is f-invariant, it is f^n-invariant. Hence we have the following corollary.

Corollary 5. *If f is a C^1 expanding map and Λ is a repeller, then*

$$s_n \leq dim_H \Lambda \leq \underline{dim}_B \Lambda \leq \overline{dim}_B \Lambda \leq t_n$$

where s_n and t_n are the unique roots of the Bowen's equations $P(f^n, -t \log \|Df^n(x)\|) = 0$ and $P(f^n, -t \log m(Df^n(x))) = 0$ respectively.

Next we prove that the sequences $\{t_{2^k}\}$ and $\{s_{2^k}\}$ are monotone.

Theorem 6.2. *The sequence $\{s_{2^k}\}$ is monotone, and*

$$\lim_{k\to\infty} s_{2^k} = s_*.$$

Then we have s_* is the unique root of equation $P^*(f, -t\{\log \|Df^n(x)\|\}) = 0$.

Proof. First we prove that the sequence $\{s_{2^n}\}$ is monotone increasing. Let $\varphi_n = -\log \|(Df^n(x)\|$ and $\mathcal{F} = \{\varphi_n\}$. Then it is a sup-additive function sequence. For a fixed $k \in \mathbb{N}$,

$$P_k(\phi, \epsilon) = \sup\{\sum_{x\in E} \exp S_n\phi(x) : E \text{ is a } (n, \epsilon) - \text{separated subset of } X\}.$$

For $\forall \epsilon > 0$, by the uniformly continuity of f, there exists $\delta > 0$ such that if $E \subset M$ is an (n, ϵ) separated set of $f^{2^{k+1}}$, then E is an $(2n, \delta)$ separated set of f^{2^k} and $\delta \to 0$ when $\epsilon \to 0$. Using the supadditivity of φ_n, the Birkhoff sum $S_n\phi_{2^{k+1}}$ of $\varphi_{2^{k+1}}$ with respect to $f^{2^{k+1}}$ has the following property:

$$\begin{aligned}
S_n\varphi_{2^{k+1}}(x) &= \varphi_{2^{k+1}}(x) + \varphi_{2^{k+1}}(f^{2^{k+1}}x) + \cdots + \varphi_{2^{k+1}}(f^{2^{k+1}(n-1)}x) \\
&\geq \varphi_{2^k}(x) + \varphi_{2^k}(f^{2^k}x) + \varphi_{2^k}(f^{2^{k+1}}x) + \varphi_{2^k}(f^{2^{k+1}}f^{2^k}x) \\
&+ \cdots + \varphi_{2^k}(f^{2^{k+1}(n-1)}x) + \varphi_{2^k}(f^{2^{k+1}(n-1)}f^{2^k}x) \\
&= S_{2n}\varphi_{2^k}(x)
\end{aligned}$$

where $S_{2n}\varphi_{2^k}(x)$ is the Birkhoff sum of φ_{2^k} with respect to f^{2^k}.

Thus

$$P_n(f^{2^{k+1}}, \varphi_{2^{k+1}}, \epsilon) \geq P_{2n}(f^{2^k}, \varphi_{2^k}, \delta).$$

Hence

$$P(f^{2^{k+1}}, \varphi_{2^{k+1}}) \geq 2P(f^{2^k}, \varphi_{2^k}).$$

Therefore if $s_{2^{k+1}}$ is the unique root of Bowen's equation $P(t\varphi_{2^{k+1}}) = 0$, then we have

$$0 = P(f^{2^{k+1}}, s_{2^{k+1}}\varphi_{2^{k+1}}) \geq 2P(f^{2^k}, s_{2^{k+1}}\varphi_{2^k}).$$

Since the function $P(f^{2^k}, t\phi_{2^k})$ is monotone decreasing, $s_{2^k} \leq s_{2^{k+1}}$.

The arbitrariness of k implies that the sequence $\{s_{2^k}\}$ monotone decreasing.

Next we prove that

$$P^*(f, \mathcal{F}) \geq \frac{1}{k}P(f^k, \varphi_k) \quad \forall k \in \mathbb{N}.$$

For a fixed $k \in \mathbb{N}$, let $n = km + r, \ \ 0 \leq r < k$, and let $C = \min_{x\in M} \max_{1\leq i\leq k} \phi_i(x)$. For $\forall\epsilon > 0$, by the uniformly continuity of f, there exists $\delta > 0$ such that if $E \subset M$ is an (n, ϵ) separated set of f, then E is an (m, δ) separated set of f^k and $\delta \to 0$ when $\epsilon \to 0$. Using the sup-additivity of φ_n, we have

$$\varphi_n(x) \geq \varphi_k(x) + \varphi_k(f^k(x)) + \cdots + \varphi_k(f^{(m-1)k}(x)) + \varphi_r(f^{mk}(x)).$$

Thus

$$P_n^*(f, \mathcal{F}, \epsilon) \geq P_m(f^k, \varphi_k, \delta) \times e^{-C}.$$

Hence

$$P^*(f, \mathcal{F}, \epsilon) \geq \frac{1}{k}P(f^k, \varphi_k, \delta).$$

It gives that

$$P^*(f, \mathcal{F}) \geq \frac{1}{k}P(f^k, \varphi_k).$$

Therefore

$$P^*(f, \mathcal{F}) \geq \frac{1}{2^k}P(f^{2^k}, \phi_{2^k}) \quad \forall k \in \mathbb{N}.$$

Let $t\mathcal{F} = \{t\phi_n(x)\}$. Then we have

$$P^*(f, s_{2^k}\mathcal{F}) \geq \frac{1}{2^k}P(f^{2^k}, s_{2^k}\phi_{2^k}) = 0 \quad \forall k \in \mathbb{N}.$$

The monotone decreasing of $P^*(f, t\mathcal{F})$ with respect to t implies that the unique root s_* of the equation

$$P^*(f, t\mathcal{F}) = 0$$

satisfies

$$s_* \geq s_{2^k} \quad \forall k \in \mathbb{N}.$$

Thus

$$s_* \geq \overline{s} = \lim_{k \to +\infty} s_{2^k}.$$

Next we want to prove that

$$\overline{s} \geq s_*.$$

For a fixed m,

$$\frac{1}{2^m} P(f^{2^m}, s_{2^m} \varphi_{2^m}) = 0$$

using the variational principle, for any $\mu \in \mathcal{M}(f) \subset \mathcal{M}(f^{2^m})$, it has

$$h_\mu(f) + \frac{1}{2^m} s_{2^m} \int_M \varphi_{2^m} d\mu = \frac{1}{2^m}(h_\mu(f^{2^m}) + s_{2^m} \int_M \varphi_{2^m} d\mu) \leq 0.$$

Let $m \to \infty$, we

$$h_\mu(f) + \overline{s} \lim_{m \to \infty} \int_M \frac{1}{2^m} \varphi_{2^m} d\mu \leq 0.$$

Using sup-additive variational principle, we have

$$P^*(f, \overline{s}\{\varphi_n\}) \leq 0.$$

Since $P(f, t\{\varphi_n\})$ is strictly monotone decreasing with respect to t, we have

$$s_* \leq \overline{s}.$$

$\square$

Lemma 6.1. *If $\phi_n(x)$ is a subadditive sequence, then*

$$\lim_{k \to \infty} \frac{1}{2^k} P(f^{2^k}, \phi_{2^k}) \leq \lim_{m \to \infty} P(f, \frac{\phi_{2^m}}{2^m}).$$

Proof. For a fixed $k \in \mathbb{N}$. It is well known that if $E \subset M$ is an (n, ϵ) separated set of f^{2^k}, then E is an $(n2^k, \epsilon)$ separated set of f. By the definition

$$\begin{aligned} P(f^{2^k}, \phi_{2^k}) &= \lim_{\epsilon \to \infty} \limsup_{n \to \infty} \frac{1}{n} \log \sup\{\sum_{x \in E} \exp(\hat{S}_n \phi_{2^k}(x)) \\ &: \ E \text{ is a } (n, \epsilon) \text{ separated set of } f^{2^k}\}, \end{aligned}$$

where

$$(\hat{S}_n \phi_{2^k}(x)) = \phi_{2^k}(x) + \phi_{2^k}(f^{2^k} x) + \cdots + \phi_{2^k}(f^{(n-1)2^k} x).$$

Hence for a fixed $m < k$, let $2^k = 2^m q + r$ and $C = \max_{x\in M}\max_{i=1,\cdots,2^m}\phi_i(x)$, the subadditivity of ϕ_n implies that

$$\begin{aligned}
\phi_{2^k}(x) &\leq \frac{1}{2^m}\sum_{j=0}^{2^m-1}\sum_{i=0}^{q-2}\phi_{2^m}(f^{i2^m+j}(x)) + \frac{1}{2^m}\sum_{j=0}^{2^m-1}[\phi_j(x)+\phi_{2^m-j+l}(f^{(q-1)2^m+j}(x))] \\
&\leq \sum_{i=0}^{2^k-1}\frac{1}{2^m}\phi_{2^m}(f^i(x)) + 4C.
\end{aligned}$$

Thus for $1 \leq j \leq n-1$, we have

$$\phi_{2^k}(f^{2^k j}(x)) \leq \sum_{i=0}^{2^k-1}\frac{1}{2^m}\phi_{2^m}(f^i(f^{2^k j}(x)) + 4C.$$

Hence

$$\begin{aligned}
\hat{S}_n\phi_{2^k}(x) &= \phi_{2^k}(x) + \phi_{2^k}(f^{2^k}x) + \cdots + \phi_{2^k}(f^{(n-1)2^k}x) \\
&\leq \sum_{i=0}^{n2^k-1}\frac{1}{2^m}\phi_{2^m}(f^i(x)) + 4nC \\
&= S_{n2^k}(\frac{1}{2^m}\phi_{2^m})(x) + 4nC.
\end{aligned}$$

It gives that

$$P_n(f^{2^k}, \phi_{2^k}, \epsilon) \leq P_{n2^k}(f, \frac{1}{2^m}\phi_{2^m}, \epsilon) \times e^{4nC}.$$

Thus

$$\begin{aligned}
P(f^{2^k}, \phi_{2^k}) &\leq 2^k P(f, \frac{1}{2^m}\phi_{2^m}) + \lim_{n\to\infty}\frac{1}{n}\log e^{4nC} \\
&= 2^k P(f, \frac{1}{2^m}\phi_{2^m}) + 4C.
\end{aligned}$$

Therefore

$$\lim_{k\to\infty}\frac{1}{2^k}P(f^{2^k}, \phi_{2^k}) \leq P(f, \frac{1}{2^m}\phi_{2^m}) \quad \text{for all } m \in Z^+.$$

Hence

$$\lim_{k\to\infty}\frac{1}{2^k}P(f^{2^k}, \phi_{2^k}) \leq \lim_{m\to\infty}P(f, \frac{1}{2^m}\phi_{2^m}).$$

□

Lemma 6.2.

$$\lim_{n\to\infty}P(f, \frac{\phi_{2^k}}{2^k}) \leq P(f, \mathcal{F}).$$

Proof. Since $f : \Lambda \to \Lambda$ is expanding map, $h_\mu(f)$ is an upper-semi continuous function from $\mathcal{M}(f|_\Lambda)$ to R. From variational principle of topological pressure [17], we have that for every $k \in Z^+$ there exists $\mu_{2^k} \in \mathcal{M}(f|_\Lambda)$ such that

$$P(f|_\Lambda, \frac{1}{2^k}\phi_{2^k}) = h_{\mu_{2^k}}(f) + \int_\Lambda \frac{1}{2^k}\phi_{2^k} d\mu_{2^k}.$$

Since $\mathcal{M}(f|_\Lambda)$ is compact, it implies that μ_{2^k} has a subsequence which converges to $\mu \in \mathcal{M}(f|_\Lambda)$. Without loss of generality, suppose that μ_{2^k} converges to μ. Using the subadditivity and invariant of μ_{2^k}, then we have for every $k \in \mathbb{N}$

$$h_{\mu_{2^k}}(f) + \int_\Lambda \frac{\phi_{2^k}(x)}{2^k} d\mu_{2^k} \leq h_{\mu_{2^k}}(f) + \int_\Lambda \phi_1(x) d\mu_{2^k}.$$

Furthermore for fixed $s \in \mathbb{N}$. If $k > s$, from the subadditivity and invariance of μ_{2^k}, it has

$$h_{\mu_{2^k}}(f) + \int_\Lambda \frac{\phi_{2^k}(x)}{2^k} d\mu_{2^k} \leq h_{\mu_{2^k}}(f) + \int_\Lambda \frac{\phi_{2^s}(x)}{2^s} d\mu_{2^k}.$$

Since $h_\mu(f)$ is a upper-semi continuous function, we have

$$\begin{aligned} \lim_{k\to\infty} P(f, \frac{\phi_{2^k}}{2^k}) &= \lim_{k\to\infty} (h_{\mu_{2^k}}(f) + \int_\Lambda \frac{\phi_{2^k}(x)}{2^k} d\mu_{2^k}) \\ &\leq \lim_{k\to\infty} (h_{\mu_{2^k}}(f) + \int_\Lambda \frac{\phi_{2^s}(x)}{2^s} d\mu_{2^k}) \\ &\leq h_\mu(f) + \int_\Lambda \frac{\phi_{2^s}(x)}{2^s} d\mu. \end{aligned}$$

Since sequence $\{\int_\Lambda \phi_n(x) d\mu\}$ is sub-additive sequence, it has

$$\lim_{n\to\infty} \int_\Lambda \frac{\phi_n(x)}{n} d\mu = \inf_{n\geq 1}\{\int_\Lambda \frac{\phi_n(x)}{n} d\mu\}.$$

The arbitrariness of $s \in \mathbb{N}$ implies that

$$\lim_{k\to\infty} P(f, \frac{\phi_{2^k}}{2^k}) \leq h_\mu(f) + \lim_{s\to\infty} \int_\Lambda \frac{\phi_{2^s}}{2^s}(x) d\mu.$$

Hence by variational principle of the sub-additive topological pressure in [4], we have

$$\lim_{k\to\infty} P(f, \frac{\phi_{2^k}}{2^k}) \leq h_\mu(f) + \lim_{s\to\infty} \int_\Lambda \frac{\phi_{2^s}}{2^s}(x) d\mu \leq P(f, \mathcal{F}).$$

This completes the proof of lemma. □

Theorem 6.3. *The sequence $\{t_{2^n}\}$ is monotone, and*

$$\lim_{n\to\infty} t_{2^n} = t^*$$

where t^* is the unique root of equation $P(f, -t\{\log m(Df^n(x))\}) = 0$.

Proof. First we prove that the sequence $\{t_{2^n}\}$ is monotone decreasing. Let $\phi_n = -\log m(Df^n(x))$. For a fixed $k \in \mathbb{N}$,

$$P_k(\phi, , \epsilon) = \sup\{\sum_{x\in E} \exp S_n\phi(x) : E \text{ is a } (n,\epsilon) - \text{separated subset of } X\}.$$

For $\forall\epsilon > 0$, by the uniformly continuity of f, there exists $\delta > 0$ such that if $E \subset M$ is an (n, ϵ) separated set of $f^{2^{k+1}}$, then E is an $(2n, \delta)$ separated set of f^{2^k} and $\delta \to 0$ when $\epsilon \to 0$. Using the subadditivity of ϕ_n, the Birkhoff sum $S_n\phi_{2^{k+1}}$ of $\phi_{2^{k+1}}$ with respect to $f^{2^{k+1}}$ has the following property:

$$\begin{aligned}
S_n\phi_{2^{k+1}}(x) &= \phi_{2^{k+1}}(x) + \phi_{2^{k+1}}(f^{2^{k+1}}x) + \cdots + \phi_{2^{k+1}}(f^{2^{k+1}(n-1)}x) \\
&\le \phi_{2^k}(x) + \phi_{2^k}(f^{2^k}x) + \phi_{2^k}(f^{2^{k+1}}x) + \phi_{2^k}(f^{2^{k+1}}f^{2^k}x) \\
&+ \cdots + \phi_{2^k}(f^{2^{k+1}(n-1)}x) + \phi_{2^k}(f^{2^{k+1}(n-1)}f^{2^k}x) \\
&= S_{2n}\phi_{2^k}(x)
\end{aligned}$$

where $S_{2n}\phi_{2^k}(x)$ is the Birkhoff sum of ϕ_{2^k} with respect to f^{2^k}.

Thus

$$P_n(f^{2^{k+1}}, \phi_{2^{k+1}}, \epsilon) \le P_{2n}(f^{2^k}, \phi_{2^k}, \delta).$$

Hence

$$P(f^{2^{k+1}}, \phi_{2^{k+1}}) \le 2P(f^{2^k}, \phi_{2^k}).$$

Therefore if $t_{2^{k+1}}$ is the unique root of Bowen's equation $P(t\phi_{2^{k+1}}) = 0$, then we have

$$0 = P(f^{2^{k+1}}, t_{2^{k+1}}\phi_{2^{k+1}}) \le 2P(f^{2^k}, t_{2^{k+1}}\phi_{2^k}).$$

The monotone decreasing of the function $P(f^{2^k}, t\phi_{2^k})$ implies that $t_{2^k} \ge t_{2^{k+1}}$.

The arbitrariness of k implies that the sequence $\{t_{2^k}\}$ monotone decreasing. Hence limit exists and we denote the limit of this sequence by $\bar{t}$. From the proof as above, we have

$$\frac{P(f^{2^{k+1}}, \phi_{2^{k+1}})}{2^{k+1}} \le \frac{P(f^{2^k}, \phi_{2^k})}{2^k} \le \cdots \le \frac{P(f^2, \phi_2)}{2} \le P(f, \phi).$$

Next we prove that

$$P(f, \mathcal{F}) \le \frac{1}{k}P(f^k, \phi_k) \quad \forall k \in \mathbb{N}.$$

For a fixed $k \in \mathbb{N}$, let $n = km + r,\ \ 0 \le r < k$, and let $C = \max_{x\in M} \max_{1\le i\le k} \phi_i(x)$. For $\forall\epsilon > 0$, by the uniformly continuity of f, there exists $\delta > 0$ such that if $E \subset M$ is an (n, ϵ) separated set of f, then E is an (m, δ) separated set of f^k and $\delta \to 0$ when $\epsilon \to 0$. Using the subadditivity of ϕ_n, we have

$$\phi_n(x) \le \phi_k(x) + \phi_k(f^k(x)) + \cdots + \phi_k(f^{(m-1)k}(x)) + \phi_r(f^{mk}(x)).$$

Thus

$$P_n(f, \mathcal{F}, \epsilon) \le P_m(f^k, \phi_k, \delta) \times e^C.$$

Hence

$$P(f, \mathcal{F}, \epsilon) \le \frac{1}{k} P(f^k, \phi_k, \delta).$$

It gives that

$$P(f, \mathcal{F}) \le \frac{1}{k} P(f^k, \phi_k).$$

Therefore

$$P(f, \mathcal{F}) \le \frac{1}{2^k} P(f^{2^k}, \phi_{2^k}) \quad \forall k \in \mathbb{N}. \tag{6.4}$$

Let $t\mathcal{F} = \{t\phi_n(x)\}$. Then we have

$$P(f, t_{2^k}\mathcal{F}) \le \frac{1}{2^k} P(f^{2^k}, t_{2^k}\phi_{2^k}) = 0 \quad \forall k \in \mathbb{N}.$$

Therefore the unique root t^* of the equation

$$P(f, t\mathcal{F}) = 0$$

satisfies

$$t^* \le t_{2^k} \quad \forall k \in \mathbb{N}.$$

Thus

$$t^* \le \bar{t} = \lim_{k\to+\infty} t_{2^k}.$$

Next we want to prove that

$$\bar{t} \le t^*.$$

From Theorem 6.2 and lemma 6.1, 6.2, we have the sequence $\{\frac{1}{2^k} P(f^{2^k}, \phi_{2^k})\}$ is monotone decreasing and it converges to $P(f, \mathcal{F})$. By the definition, it is easy to prove that

$$0 \le \frac{P(f^{2^k}, \bar{t}\phi_{2^k})}{2^k} - \frac{P(f^{2^k}, t_{2^k}\phi_{2^k})}{2^k} \le |\bar{t} - t_{2^k}|C, \quad \forall k \in \mathbb{N},$$

where $C = \max_{x\in M} |\phi_1(x)|$. Let $k \to \infty$, we have

$$P(f, \bar{t}\mathcal{F}) = 0.$$

Hence it has,

$$\bar{t} = t^*.$$

□

Theorem 6.4. $t^* = s_*$.

Proof. From theorems as above, we have functions

$$P(f, -t\{\log m(Df^n(x))\})$$

and

$$P(f, -t\{\log \|Df^n(x)\|\})$$

coincide and both of them have unique zero points. Therefore

$$t^* = s_*.$$

□

The proof of main theorem:

From Corollary 5 and theorems 6.4 as above, we have

$$dim_H\Lambda = \underline{dim}_B\Lambda = \overline{dim}_B = s_* = t^*.$$

This completes the proof of main theorem.

Corollary 6. *If Λ be an average conformal repeller, then the Hausdorff dimension of Λ is zero t^* of*

$$t \mapsto P(-t\frac{1}{d}\log(|det(Df)|)),$$

where $d = dimM$ and $t \mapsto P(-t\frac{1}{d}\log(|det|Df|))$ is classical topological pressure.

Proof. If Λ be an average conformal repeller, then by Theorem 4.2, we have

$$\lim_{n\to\infty}\frac{1}{n}(\log\|Df^n(x)\| - \log m(Df^n(x))) = 0$$

uniformly on Λ.

On the other hand, $\log(m(Df^n(x)) \leq \frac{1}{d}\log(|det(Df^n(x))|) \leq \log(|Df^n(x)|)$. Therefore

$$P(f, -t^*\{\log m(Df^n(x))\}) = P(f, -t^*\{\frac{1}{d}\log|det(Df^n(x))|\}) = P(f, -t^*\{\log\|Df^n(x)\|\}) = 0.$$

The additivity of $\{\log\|Df^n(x)\|\}$ implies that

$$P(f, -t^*\{\frac{1}{d}\log|det(Df^n(x))|\}) = P(f, -t^*\log\frac{1}{d}|det(Df(x)|) = 0.$$

That is say that t^* is the root of equation $P(-t\frac{1}{d}\log|det(Df)|) = 0$. This gives the proof of corollary.

□

Acknowledgement. Authors would like to thank Prof.Dejun Feng, Prof.Y.Pesin and Prof.Marcelo Viana for their discussions and suggestions. Authors also would like to thank referees for their suggestions. Ban is partially supported by the National Science Council, R.O.C.(Contract No. NSC 95-2115-M-026-003) and the National Center for Theoretical Sciences, and Cao is partially supported by by NSFC(10571130), NCET, and 973 Project (2007CB814800).

References

[1] Barreira, L.: Dimension estimates in nonconformal hyperbolic dynamics. Nonlinearity 16 (2003), no. 5, 1657–1672.

[2] Barreira, L.: A non-additive thermodynamic formalism and applications to dimension theory of hyperbolic dynamical systems Ergodic Theory Dyn. Syst. 16(1996) 871–927.

[3] Bowen, R.: Hausdorff dimension of quasi-circles. Inst. Hautes Études Sci. Publ. Math. 50(1979) 259–73.

[4] Cao Yongluo, Feng Dejun, Huang Wen: The thermodynamic formalism for submultiplicative potentials. Discrete and Continuous Dynamical Systems 20 (2008), no. 3, 639C657.

[5] Douady, A., Oesterl, J.: Dimension de Hausdorff des attracteurs. C. R. Acad. Sci. Paris 290(1980) 1135–8.

[6] Falconer, K.: The Hausdorff dimension of self-affine fractals. Math. Proc. Camb. Phil. Soc. 103(1988) 339–50.

[7] Falconer, K.: Dimensions and measures of quasi self-similar sets. Proc. Am. Math. Soc. 106(1989) 543–54.

[8] Falconer, K.: Bounded distortion and dimension for non-conformal repellers. Math. Proc. Camb. Phil. Soc. 115(1994) 315–34.

[9] Gatzouras,D., Y. Peres, Y.: Invariant measures of full dimension for some expanding maps. Ergod. Th. & Dynam. Sys. 17 (1997), 147–167.

[10] Hu Huyi: Dimensions of invariant sets of expanding maps. Commum.Math.Phys. 176(1996), 307-320.

[11] Ledrappier, F. Some relations between Dimension and Lyapunov exponents. Commum.Math.Phys. 81(1981), 229-238.

[12] Pesin, Y.: Dimension Theory in Dynamical Systems. Contemporary Views and Application (Chicago, IL: University of Chicago Press), 1997.

[13] Ruelle, D.: Repellers for real analytic maps. Ergodic Theory Dyn. Syst. 2(1982) 99–107.

[14] Ruelle, D.: An inequality for the entropy of differential maps. Bol.Soc.Bras.De Mat. 9(1978) 83-87.

[15] Simon,K.: Hausdorff dimension of hyperbolic attractors in R^3. Fractal geometry and stochastics III, 79C92, Progr. Probab., 57, Birkhauser, Basel (2004).

[16] Simon, K., Solomyak, B.: Hausdorff dimension for horseshoes in $\mathbb{R}^3$. Ergodic Theory Dyn. Syst. 19(1999) 1343–63.

[17] Walters, P.: An introduction to ergodic theory. Berlin, Springer (1982).

[18] Zhang Yingjie: Dynamical upper bounds for Hausdorff dimension of invariant sets. Ergodic Theory Dynam. Systems 17 (1997), no. 3, 739–756.